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Gauge and Space-Time Symmetry Unification

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Unification ideas suggest an integral treatment of fermion and boson spin and gauge-group degrees of freedom. Hence, a generalized quantum field equation, based on Dirac's, is proposed and investigated which contains gauge and flavor symmetries, determines vector gauge field and fermion solution representations, and fixes their mode of interaction. The simplest extension of the theory with a 6-dimensional Clifford algebra has an $SU(2)_L \times U(1)$ symmetry, which is associated with the isospin and the hypercharge, their vector carriers, two-flavor charged and chargeless leptons, and scalar particles. A mass term produces breaking of the symmetry to an electromagnetic $U(1)$, and a Weinberg's angle θ_W with $sin^2(\theta_W) = 0.25$. A more realistic $8D$ extension gives coupling constants of the respective groups $g = 1/\sqrt{2} \approx .707$ and $g' = 1/\sqrt{6} \approx .408$, with the same θ_W.

1. INTRODUCTION

Unification has proved to be a powerful assumption leading to new connections among phenomena previously considered unrelated. It is not exaggerated to say that most substantial advances in the history of physics have been accompanied by the realization of links among facts originally appearing to be independent. The application of unification ideas differs, however, from one case to another in scope, methods, and results, and it is therefore difficult to characterize it uniquely by a single rule. Thus, the unification of known facts has sometimes led to the prediction of new phenomena, and these connections have been either experimental, theoretical, or both. It shall be useful instead to briefly review some highlights.

The concept of unification is linked to the very idea of science (or as then considered philosophy) conceived by the early Greek philosophers, who in their research into nature sought unifying principles, although those they

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found were premature in their applicability. However, a perdurable idea from those times representing probably the most powerful tool in physics is that of assuming a mathematical structure behind physical phenomena, an idea ascribed to Pythagoras. In modern times Galileo helped to revive the idea of the universality of physical law in the cosmos by presenting evidence (e.g., the shadows engendered by the sun on the moon). The principle of relativity is a related idea he discovered, which assumes this universality for different inertial frames, putting powerful constraints on possible allowed laws. Newton showed, with his new understanding of gravity, that the motion of cosmic and terrestrial bodies obeys the same laws, thus demonstrating for the first time a deep relation between phenomena in both realms.

Electric and magnetic phenomena were considered separated until the 19th century. With the work of Ampère and Faraday it was found experimentally that one leads to the other by changing the kinematic state of the charges involved. Maxwell carried out the formalization of this into a series of equations which provided a new understanding of light as one of many possible waves of electromagnetic origin, and traveling at a speed that was predicted from the equations.

In the 20th century, Einstein's special relativity integrated Galileo's relativity principle with the invariance of Maxwell's equations into a new framework by dethroning time from its privileged place and putting it on a similar footing to space, while the speed of light was assumed constant in all reference frames. New phenomena were predicted, such as the equivalence between mass and energy. These ideas were expanded by linking gravity, matter, and space-time through general relativity (GR), a theory which assumes a geometrical framework. However, this was done only partially, since in Einstein's GR equations only the side describing gravity and space-time's geometry has this interpretation, while the other, containing the energy-momentum tensor, does not necessarily have this form, and awaits geometrization. GR predicts new phenomena such as black holes, while in the Newtonian weak gravitational limit it produces small corrections.

Einstein attempted unsuccessfully to unify gravity and electromagnetism; in the meantime Kaluza and Klein developed the idea of extending GR to more than $3 + 1$ dimensions, relating an additional dimension to a vector potential which could be shown to describe the electromagnetic field. This work has not led to more information, the prediction of new phenomena, or testable options, but does represent a viable possibility. Therefore, although it cannot be classified as a successful unification, it retains the status of a useful working hypothesis that is actually applied in theories such as supergravity or superstrings.

Quantum mechanics (QM) successfully accounted for the prevalence of particle and wave characteristics encountered in different experiments with the same objects, which would be contradictory in a classical framework. This comprised a completely new feature for the constituents of nature, which were previously thought to belong to separate classes presenting different kinds of behavior. The introduction of Planck's constant required by QM gives rise, when using Newton's gravity constant and the speed of light, to fundamental values of mass, time, and position; this constitutes a unification in the sense that all measurable quantities can be related to these fundamental constants. A theory which would join together GR and QM should certainly use these.

One of the most beautiful examples of unification comes from Dirac, who discovered a new type of equation that satisfied both the principles of special relativity and those of QM. Their marriage in this new setting provided a new understanding of the spin-1/2 degree of freedom, a variable previously postulated to account for various atomic phenomena and understood to be related to magnetic properties of fermions, but with an otherwise inscrutable origin. Dirac's equation not only naturally gives rise to this variable, but also predicts with relatively close accuracy the electron's magnetic moment.

In more recent times unification ideas have been successful in relating the weak and electromagnetic interactions in the Weinberg–Salam model [5, 7, 8], which considers them to originate in a gauge symmetry, although their respective groups $SU(2)_L \times U(1)$ assume totally different forms. Still, the theory succeeded in predicting parameters such as the masses of the vectors carrying these interactions and the existence of neutral currents.

Many of today's puzzles in fundamental physics are encountered in the current theory of elementary particles and fields, the standard model (SM). Although it is quite successful in describing their behavior, its very construction requires input determined by phenomenology, but which is otherwise ad hoc and consists of a large number of parameters. Worse, many aspects of this input still need justification. It is not clear why there are three generations of leptons and quarks nor is the origin of their masses and the latter's mixing angles. Neither is it clear what is the source of the parameters needed to describe the Higgs particle, which is as yet only a mathematical device to break the gauge symmetry and give masses to particles; indeed, we lack a more fundamental reason for the presence of a spin-0 particle. We also lack information on the origin of the gauge groups of the fundamental interactions $SU(3) \times SU(2)_L \times U(1)$, the origin of their coupling constant values, and the reason the isospin force acts only on a given chirality, which leads to parity violation [6]. However, in this case, very interesting connections have been obtained from grand unified models on both the forces and the values of coupling constants [4]. These models assume a common origin for these forces' gauge groups through the postulation of a group containing them as subgroups. Still, the overall picture hints at a missing piece of information on an underlying principle. It may be worth returning to unification ideas for a clue. In particular, we now concentrate on the current concepts of spin and space-time symmetries and follow a possible connection path to gauge symmetries.

While QM offers a common description for the shared properties of bosons and fermions, it still requires a specialized treatment for each to account for their differences. Thus, while the space-time description of the propagation of a fermion is similar to that of a boson, this differs in the spin wave functions, which from quantum field theory (QFT) are known to have a determinant influence on their very different collective behavior. A unified theory describing both kinds of particles should address the question of their spin. The only physical connection comes through the vertex interaction, which is determined uniquely by the gauge symmetry (e.g., in the electromagnetic case). Still, boson and fermion degrees of freedom are presently otherwise assumed independent from each other. In looking for a closer connection between them it is worth having in mind that the spin-1/2 particle representation of the Lorentz group $SO(3, 1)$ is more fundamental than the vector one, as the latter can be obtained from a tensor product of the first, but not the other way around.

On another plane, the fact that a particle description requires both configuration (or momentum) and spin spaces leads in turn to the fact that it is only a combination of both types of corresponding generators that allows for invariance under Lorentz transformations, which makes them equally necessary. In this context, it is worth recalling the Kaluza–Klein idea and wonder whether there exists a connection of the forces of nature to extended spin spaces instead of additional spatial dimensions. In a way, this idea underlay the attempt of Heisenberg and Condon to understand the difference between the proton and the neutron. Having in mind the similarity with spin, they assigned them with hindsight a doublet structure, calling it isotopic spin or isospin, a concept that evolved into the $SU(2)_L$ group underlying the modern treatment of the weak interactions.

In this paper we propose a new field equation, based on Dirac's, which allows for a unified treatment of both boson and fermion spin degrees of freedom by making the solutions share the same solution space and at the same time which encompasses degrees of freedom which can be assigned to the gauge groups. The equation and the surrounding formalism are developed in a quantum mechanical relativistic framework, but some aspects of QFT will be touched upon. We will show that the dimensionality of the solution space restricts both the possible solutions and the symmetries present, and that from these an interaction prescription emerges naturally among the field solutions. In particular, we obtain vertices and their coupling constants. We analyze the simplest extension to $5 + 1$ and we find that an $SU(2)_L \times U(1)$

symmetry is predicted. The solutions will be related to physical fields. In Section 2 we study the $(3 + 1)$-dimensional version of the new equation by considering its symmetries and a set of commuting operators characterizing the solutions. We also find and analyze their link to quantized fields. In Section 3 we present its boson solutions, both at the massless and massive levels, and in Section 4 study a particular reduction of the equation and its transformations leading to fermion solutions too. We argue that both versions of the field equation contain a gauge invariance. In Section 5 we present some conserved currents and through them we find a link to a vertex interaction between a pair of spin-1/2 particles and a boson, which is implied in the formalism. In Section 6 we generalize the equation to six dimensions using the $5 + 1$ Clifford algebra and we analyze the embedded 4D Clifford subalgebras and corresponding symmetries. We show that for one subalgebra chain an $SU(2)_L \times U(1)$ symmetry is implied. In Section 7 we present the massless solutions and link these symmetries to the isospin and hypercharge generators, respectively. In Section 8 we present the massive ones. In Section 9 we link these solutions to physical fields in the SM and obtain the fermion–vector couplings and coupling constants. In Section 10 we summarize this work, indicate its main results, and draw conclusions.

2. GENERALIZED FIELD EQUATION FROM DIRAC FORMALISM

We search for a description of vectors and scalars as close as possible to the one that exists for fermions in order to be able to relate both representations. We also demand that the field equation which provides this description be enclosed in a variational principle framework. Indeed, these requirements are achieved by generalizing Dirac's equation and extending its multiplet content. At this point we concentrate only on the free particle case, and later we show how interactions are implied in this formalism. Then, instead of assuming the Dirac operator acts on a spinor [3]

$$(i\partial_\mu \gamma^\mu - M)\psi = 0 \tag{1}$$

where ψ is the column vector with components ψ_α, we assume it acts on a 4×4 matrix Ψ with components $\Psi_{\alpha\beta}$ so that the equation becomes

$$(i\partial_\mu \gamma^\mu - M)\Psi = 0 \tag{2}$$

The form of this equation implies all symmetry operators valid for the Dirac equation (1) (with its corresponding particular massless and massive cases) will be valid as well for it. The operators therefore satisfy the Poincaré algebra. There are other possible Lorentz-invariant terms that could enter Eq.

(2); further justification for the choice of the terms in this equation is related to a gauge symmetry described in Section 4.

We postulate that all transformations and symmetry operations on the Dirac operator $(i\partial_\mu\gamma^\mu - M) \to U(i\partial_\mu\gamma^\mu - M)U^{-1}$ induce a corresponding transformation

$$\Psi \to U\Psi U^\dagger \tag{3}$$

Here the lhs U is fixed by the Dirac operator transformations, but there is a liberty for the rhs term, the choice of which $U^\dagger$ will shortly prove its utility. With this assumption the elements of Ψ, which can be expanded in terms of the tensor product of two spinors $\sum_{i,j} a_{ij}|\omega_i\rangle\langle\omega_j|$, are expected to Lorentz-transform as scalars, vectors, and antisymmetric tensors. We will show that modified symmetry operators also classify some solutions as fermions.

The vector space spanned by the matrix solutions allows one to define an algebra to which they belong. By using the matrix product, if A, B are solutions, we find the new field

$$C = AB \tag{4}$$

is another element of the algebra which may or not be a solution, but lives in the same vector space. We find here a connection to QFT as we have an algebra of operator solutions. In fact, we will show that the product among fields leads to interactions among them.

The quantum mechanical dot product of A, B is defined by

$$\langle A|B\rangle = \mathrm{tr}(A^\dagger B) \tag{5}$$

A trace of over the coordinates is also implied. This definition satisfies the usual properties expected for a measure. The use of the product in Eq. (4) implies the number of terms entering the point product is not restrained and it may include more than two fields to be evaluated. Expectation values of operators or any matrix element with the overlap of two solutions can therefore be defined. An interpretation of these products also requires taking care of the Lorentz structure.

We note transformation (3) is also valid for Hermitian conjugated fields $\Psi^\dagger$ which satisfy the equation

$$0 = \Psi^\dagger(-i\overleftarrow{\partial}_\mu\gamma^{\mu\dagger} - M) \tag{6}$$

We will extend our space of solutions by considering also combinations of fields A, $B^\dagger$,

$$A + B^\dagger \tag{7}$$

respectively satisfying Eqs. (2) and (6). It is by taking account of these fields that we can span the function space on the 32-dimensional complex 4×4 matrices.

2.1. Conserved Operators

We shall be interested in plane-wave solutions of the form

$$\Psi_k^{(+)}(x) = u(k)\, e^{-ikx} \tag{8}$$

$$\Psi_k^{(-)}(x) = v(k)\, e^{ikx} \tag{9}$$

where k^μ is the momentum four-vector $(E, \mathbf{k})$, $k_0 = E$.

By putting Eq. (2) into Hamiltonian form and using the plane-wave states of Eqs. (8) and (9), we find that each spinor satisfies the stationary equation

$$\gamma_0(\mathbf{k} \cdot \boldsymbol{\gamma} + M)u(k) = Eu(k) \tag{10}$$

and

$$\gamma_0(-\mathbf{k} \cdot \boldsymbol{\gamma} + M)v(k) = -Ev(k) \tag{11}$$

with $\boldsymbol{\gamma} = (\gamma^1, \gamma^2, \gamma^3)$. To classify the solutions, we use the Hamiltonian

$$H = \gamma_0(\mathbf{k} \cdot \gamma + M) \tag{12}$$

and the Pauli–Lubansky vector

$$W_\mu = -\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma} J^{\nu\rho} p^\sigma \tag{13}$$

constructed from the Lorentz-transformation generators

$$J_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) + \tfrac{1}{2}\sigma_{\mu\nu} \tag{14}$$

with the spin operators given by

$$\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu] \tag{15}$$

and momentum operator

$$p^\mu = i\partial^\mu \tag{16}$$

W_μ is projected over the spacelike four-vector n_k, orthogonal to the momentum, of norm -1 (the conventions for the norm $g_{\mu\nu}$ are given in the Appendix)

$$n_k = \left(\frac{|\mathbf{k}|}{M}, \frac{E\mathbf{k}}{M|\mathbf{k}|}\right) \tag{17}$$

giving

$$\frac{1}{M} W \cdot n_k = \mathbf{\Sigma} \cdot \hat{\mathbf{k}} \tag{18}$$

where

$$\mathbf{\Sigma} = \tfrac{1}{2}\gamma_5\gamma_0\boldsymbol{\gamma} \tag{19}$$

The definition in Eq. (18) is valid both for the massless and the massive cases.

2.2. Solutions as Quantized Fields

Consistency of the definition (3) when applied to the generator of time translations, the Hamiltonian, implies formally that the energy should be obtained by taking the commutator

$$[\gamma_0(-i\nabla \cdot \boldsymbol{\gamma} + M), \Psi] \tag{20}$$

This operation calls for a rule on the action of the derivative on the right. We will proceed heuristically here and apply the transformation rule $p^\mu \rightarrow k^\mu$ on H. We apply the same rule on the $(1/M)W \cdot n_k$ operator, which leads to $\mathbf{\Sigma} \cdot \hat{\mathbf{k}}$. This prescription is already taken into account in Eqs. (12) and (18) and is as expected for spin-derivative operators acting as a tensor-product space. We shall use this assignment for these operators, which classify the solutions throughout this paper. As a bonus, we obtain that Hermitian conjugates of negative-energy solutions have positive energies with the opposite spin s, just as occurs in QFT, which in turn reproduces hole theory. Indeed, assuming for the $v(k)$ component of $\Psi_k^{(-)}(x)$ in Eq. (9),

$$[H, v(k)] = -Ev(k) \tag{21}$$

$$[-\mathbf{\Sigma} \cdot \hat{\mathbf{k}}, v(k)] = sv(k) \tag{22}$$

we find that for the Hermitian conjugate wave function field $v^\dagger(k)e^{-ikx}$, satisfying Eq. (6),

$$[H, v^\dagger(k)] = Ev^\dagger(k) \tag{23}$$

$$[-\mathbf{\Sigma} \cdot \hat{\mathbf{k}}, v^\dagger(k)] = -sv^\dagger(k) \tag{24}$$

We expect a more formal justification of this operation will be given in the rigorous context of QFT. In addition, consistency with Eq. (2) will require a choice of the normalization for agreement with the energy E.

Table I. $V - A$ Terms

Vector solutions	$\gamma_0\gamma^3$	$\frac{i}{2}\gamma_1\gamma_2$	$[H/k_0, \cdot]$	$[\mathbf{\Sigma} \cdot \hat{\mathbf{p}}, \cdot]$
$u_{-1}(k) = \frac{1}{4}(1 - \gamma_5)\gamma_0(\gamma_1 - i\gamma_2)$	1	−1/2	2	−1
$u_{-1}(\tilde{k}) = \frac{1}{4}(1 - \gamma_5)\gamma_0(\gamma_1 + i\gamma_2)$	−1	1/2	2	−1
$u_0(k) = \frac{1}{4}(1 - \gamma_5)\gamma_0(\gamma_0 - \gamma_3)$	1	−1/2	0	0
$u_0(\tilde{k}) = \frac{1}{4}(1 - \gamma_5)\gamma_0(\gamma_0 + \gamma_3)$	−1	1/2	0	0

3. VECTORS, SCALARS, AND ANTISYMMETRIC TENSORS

3.1. Massless Solutions

The massless equation

$$i\partial_\mu\gamma^\mu\Psi = 0 \tag{25}$$

leads to expressions for the operators in Eqs. (12) and (18), assuming (from here and for all massless solutions, except when otherwise stated) that the space component of momentum k^μ is along the $\hat{\mathbf{z}}$ direction,

$$\mathbf{\Sigma} \cdot \hat{\mathbf{k}} = \frac{i}{2}\gamma_1\gamma_2 \tag{26}$$

$$H/k_0 = \gamma_0\gamma^3 \tag{27}$$

where the former is the helicity operator and latter is the Hamiltonian divided by the energy.

The polarization components of the solutions of Eq. (25), bilinear in the γs, are given on Tables I and II. We set the coordinate dependence as

$$\Psi_{ki}^{(+)V-A}(x) = u_i(k)e^{-ikx} \tag{28}$$

$$\Psi_{ki}^{(+)V+A}(x) = \tilde{u}_i(k)e^{-ikx} \tag{29}$$

They are given together with their quantum numbers corresponding to the operators in Eqs. (26) and (27). The solutions are also eigenfunctions of these

Table II. $V + A$ Terms

Vector solutions	$\gamma_0\gamma^3$	$\frac{1}{2}\gamma_1\gamma_2$	$[H/k_0, \cdot]$	$[\mathbf{\Sigma} \cdot \hat{\mathbf{p}}, \cdot]$
$\tilde{u}_1(k) = \frac{1}{4}(1 + \gamma_5)\gamma_0(\gamma_1 + i\gamma_2)$	1	1/2	2	1
$\tilde{u}_1(\tilde{k}) = \frac{1}{4}(1 + \gamma_5)\gamma_0(\gamma_1 - i\gamma_2)$	−1	−1/2	2	1
$\tilde{u}_0(k) = \frac{1}{4}(1 + \gamma_5)\gamma_0(\gamma_0 - \gamma_3)$	1	1/2	0	0
$\tilde{u}_0(\tilde{k}) = \frac{1}{4}(1 + \gamma_5)\gamma_0(\gamma_0 + \gamma_3)$	−1	−1/2	0	0

operators O in the simple form $O_{u_i}(k) = \lambda_{u_i}(k)$ and we present the eigenvalues λ too, where here and throughout the solutions are normalized as, e.g.,

$$\mathrm{tr}(\tilde{u}_i^\dagger(k)\tilde{u}_i(k)) = 1. \tag{30}$$

Solutions $u_{-1}(k)$, $\tilde{u}_1(k)$ correspond to on-shell particles with transverse polarizations but opposite helicities, while the off-shell $u_0(k)$, $\tilde{u}_0(k)$ are polarized in the longitudinal-scalar directions. All these solutions correspond to waves propagating in the $\hat{\mathbf{z}}$ direction. The other terms propagate in the $-\hat{\mathbf{z}}$ direction, which is denoted through the four-vector $\tilde{k}^\mu = k_\mu$ and they are classified according to the appropriate relations as Eqs. (26) and (27). The coordinate dependence of these solutions is given by

$$\Psi_{ki}^{(+)}(x) = u_i(\tilde{k})\, e^{-i\tilde{k}x} \tag{31}$$

These solutions do not represent independent polarization components as, e.g., $u_i(\tilde{k})$ can be obtained by rotating the $u_i(k)$. The classification $V + A$ and $V - A$, consisting respectively of the $\tilde{u}_i$ and u_i terms, corresponds to specifying the weight of vector and axial components, which is further clarified below. These two types of solutions are also characterized by the two vector spaces projected by $\frac{1}{2}(1 + \gamma_5)$ and $\frac{1}{2}(1 - \gamma_5)$ which they generate, respectively, but which they do not exhaust. We need to consider the negative-energy solutions

$$\Psi_{ki}^{(-)}(x) = \nu_i(k)\, e^{ikx} \tag{32}$$

and use their Hermitian conjugates, which in fact generate other polarization components, in order to completely span the space. In the massless case we have negative-energy solutions $\nu_i(k) = u_i(k)$, $\tilde{\nu}_i(k) = \tilde{u}_i(k)$ (and $\tilde{k}$ terms), that is, with opposite helicities. The combinations of the type (7), $(1/2)[\tilde{u}_i(k) \pm \tilde{\nu}_i^\dagger(k)]\, e^{-ikx}$, $(1/2)[u_i(k) \pm \nu_i^\dagger(k)]\, e^{-ikx}$ $\{\nu_i^\dagger(k) \equiv [\nu_i(k)]^\dagger\}$ will be interpreted as vector solutions with varied polarizations. The chirality operator γ_5 further characterizes these solutions as nonchiral since, using rule (3), it gives $[\gamma_5, \Psi] = 0$. The most general form of the solutions can be obtained by rotating and boosting these solutions through a Lorentz transformation, using $J_{\mu\nu}$ in Eq. (14).

Equation (2) also satisfies the discrete invariances of time and space inversion and charge conjugation, expressed respectively by the operators

$$T = i\gamma_1\gamma_3\mathcal{K}\mathcal{T} \tag{33}$$

$$P = \gamma_0\mathcal{P} \tag{34}$$

$$C = i\gamma_2\mathcal{K} \tag{35}$$

where $\mathcal{K}$ is the complex conjugation operator, $\mathcal{K}i\mathcal{K} = -i$, $\mathcal{T}$ changes $t \rightarrow$

$-t$, and $\wp$ changes $\mathbf{x} \to -\mathbf{x}$ and consequently $\mathbf{p} \to -\mathbf{p}$; we use the Dirac representation for the γ_μ matrices (see Appendix). It is then possible to form combinations of the above solutions transforming as vectors and as axial vectors. For example, the combination

$$\Psi_{k\hat{x}} = \frac{i}{2}\,[\tilde{u}_1(k) + u_{-1}(k) - \tilde{v}_1^\dagger(k) - v_{-1}^\dagger(k)]\,e^{-ikx} = \frac{i}{2}\,\gamma_0\gamma_1\,e^{-ikx} \quad (36)$$

represents a vector particle linearly polarized along $\hat{\mathbf{x}}$, that is, it transforms into $-\Psi_{k\hat{x}}(\tilde{x})$ under P with $\tilde{x}_\mu = x^\mu$. In general

$$A_\mu(x) = \frac{i}{2}\,\gamma_0\gamma_\mu\,e^{-ikx} \quad (37)$$

(and the corresponding negative-energy solution) transforms into $A^\mu(\tilde{x})$ under P, into $A^\mu(-\tilde{x})$ under T, and into $-A_\mu(-x)$ under C. We have that

$$A_{5\mu}(x) = \frac{i}{2}\,\gamma_5\gamma_0\gamma_\mu\,e^{-ikx} \quad (38)$$

transforms into $-A_5^\mu(\tilde{x})$ under P, into $A^\mu(-\tilde{x})$ under T, and into $A_{5\mu}(-x)$ under C. The combination $A_\mu(x) + CA_\mu(x)C^\dagger$ transforms into minus itself under charge conjugation, as expected for a non-axial vector. Given the quantum numbers of $A_\mu(x)$, it becomes possible to relate it to the vector potential of an electromagnetic field. Indeed, similar mixtures of $\tilde{u}$, $\tilde{v}^\dagger$, u, and $v^\dagger$ solutions have been shown, under certain conditions, to satisfy Maxwell's equations [1].

The remaining eight degrees of freedom in the massless case are classified into six forming an antisymmetric tensor and two scalars, which appear mixed in the solutions. The chirality γ_5 further divides them into left- and right-handed. Their respective coordinate dependences are

$$\Psi_{ki}^{(+)-}(x) = w_i(k)\,e^{-ikx} \quad (39)$$

$$\Psi_{ki}^{(+)+}(x) = \tilde{w}_i(k)\,e^{-ikx} \quad (40)$$

(and corresponding definitions for $\tilde{k}$) and the explicit form of the matrix components together with their quantum numbers is shown in Tables III and IV.

To see that these terms have this interpretation, we should apply transformation (3) with U containing a Lorentz transformation, acting on $1\Psi = \gamma_0\gamma_0\Psi$, which leads to $U^\dagger\gamma_0 = \gamma_0 U^{-1}$. Labeling the antisymmetric terms by

$$A_{\mu\nu} = \tfrac{1}{4}\gamma_0[\gamma_\mu, \gamma_\nu] \quad (41)$$

and the scalar and pseudoscalar terms by

Table III. Left-handed Bosons

Scalars and antisymmetric tensors	$\gamma_0\gamma^3$	$\frac{i}{2}\gamma_1\gamma_2$	$[H/k_0.\ \cdot]$	$[\boldsymbol{\Sigma}\cdot\hat{\mathbf{p}},\ \cdot]$
$w_0(k)=\frac{1}{4}(1-\gamma_5)(\gamma_0+\gamma_3)$	1	−1/2	2	0
$w_0(\tilde{k})=\frac{1}{4}(1-\gamma_5)(\gamma_0-\gamma_3)$	−1	1/2	2	0
$w_{-1}(k)=\frac{1}{4}(1-\gamma_5)(\gamma_1-i\gamma_2)$	1	−1/2	0	−1
$w_{-1}(\tilde{k})=\frac{1}{4}(1-\gamma_5)(\gamma_1+i\gamma_2)$	−1	1/2	0	−1

$$\phi = \frac{1}{2}\gamma_0 \tag{42}$$

$$\phi_5 = \frac{1}{2}\gamma_0\gamma_5 \tag{43}$$

we can write the expressions in Tables III and IV in terms of $A_{\mu\nu}$, ϕ, and ϕ_5. This requires also Hermitian conjugates of negative-energy solutions

$$\Psi_{ki}^{(-)-}(x) = z_i(k)e^{ikx} \tag{44}$$

$$\Psi_{ki}^{(-)+}(x) = \tilde{z}_i(k)e^{ikx} \tag{45}$$

where $z_i(k) = w_i(k)$, $\tilde{z}_i(k) = \tilde{w}_i(k)$ (and $\tilde{k}$ terms). While the scalar and pseudoscalar particles obtained have a straightforward interpretation as on-shell particles, the antisymmetric solutions do not have a recognizable interpretation, given that their on-shell components do not have transverse polarizations. A vector interpretation can be given using the identities

$$\begin{aligned} \tilde{w}_i(k) &= \frac{1}{2|\mathbf{k}|}\, k u_{-i}(\tilde{k}), \qquad \tilde{w}_i(\tilde{k}) = \frac{1}{2|\mathbf{k}|}\, \tilde{k} u_{-i}(k) \\ \tilde{w}_{-i}(k) &= \frac{1}{2|\mathbf{k}|}\, k \tilde{u}_i(\tilde{k}), \qquad \tilde{w}_{-i}(\tilde{k}) = \frac{1}{2|\mathbf{k}|}\, \tilde{k} \tilde{u}_i(k) \end{aligned} \tag{46}$$

and similar expressions for negative-energy solutions. The gauge symmetry discussed below suggests that some of these solutions may be gauged out.

3.2. Polarization Vectors

The solutions presented so far in Tables I–IV are given in terms of components that are eigenstates of the helicity operator and are therefore

Table IV. Right-handed Bosons

Scalars and antisymmetric tensors	$\gamma_0\gamma^3$	$\frac{i}{2}\gamma_1\gamma_2$	$[H/k_0.\ \cdot]$	$[\boldsymbol{\Sigma}\cdot\hat{\mathbf{p}},\ \cdot]$
$\tilde{w}_0(k)=\frac{1}{4}(1+\gamma_5)(\gamma_0+\gamma_3)$	1	1/2	2	0
$\tilde{w}_0(\tilde{k})=\frac{1}{4}(1+\gamma_5)(\gamma_0-\gamma_3)$	−1	−1/2	2	0
$\tilde{w}_1(k)=\frac{1}{4}(1+\gamma_5)(\gamma_1+i\gamma_2)$	1	1/2	0	1
$\tilde{w}_1(\tilde{k})=\frac{1}{4}(1+\gamma_5)(\gamma_1-i\gamma_2)$	−1	−1/2	0	1

components of spherical harmonic vectors. In general, we can show the solutions generate a quadrivector basis whose components can be given in a spherical or in a vector basis.

A set of corresponding polarization vectors $\epsilon^{(\lambda)}(k)$ can be defined which coincide with the directions taken by some of the actual solutions in Tables I–IV. We define a unitary vector n along the time direction, that is, $n^2 = 1$. Assuming a general k, we choose $\epsilon^{(1)}(k)$ and $\epsilon^{(2)}(k)$ in the transverse directions, orthogonal to k and n, and $\epsilon^{(\lambda)}(k) \cdot \epsilon^{(\lambda')}(k) = -\delta^{\lambda\lambda'}$. Then we pick $\epsilon^{(3)}(k)$, the longitudinal vector, along the plane k–n and orthogonal to n, and $\epsilon^{(0)}(k)$, the scalar component along n. These vectors are orthogonal among themselves:

$$\epsilon^{(\lambda)}(k) \cdot \epsilon^{(\lambda')}(k) = g^{\lambda\lambda'} \tag{47}$$

In the case of the solutions $\tilde{u}_i$ in Table II which propagate along $\hat{\mathbf{z}}$, the polarization vectors in the spherical basis are

$$e^{(1)}(k) = \tilde{u}_1(k) \tag{48}$$

$$e^{(2)}(k) = \tilde{u}_1(\tilde{k}) \tag{49}$$

$$e^{(3)}(k) = \frac{1}{\sqrt{2}}\,(\tilde{u}_0(\tilde{k}) - u_0(k)) \tag{50}$$

$$e^{(0)}(k) = \frac{1}{\sqrt{2}}\,(\tilde{u}_0(\tilde{k}) + u_0(k)) \tag{51}$$

The associated vector form of the polarizations is given by $\frac{1}{2\sqrt{2}}(1 + \gamma_5)\gamma_0\gamma_\mu$. In fact, the 16 components of the four vectors $\epsilon^{(\lambda)}(k)$ form a tensor which connects the two bases. The components are obtained from

$$\epsilon_\mu^{(\lambda)} = \mathrm{tr}(e*^{(\lambda)}(k)\tfrac{1}{2}\gamma_0\gamma_\mu) \tag{52}$$

where we use the conjugate polarizations

$$e*^{(1)}(k) = \frac{1}{4}(1 + \gamma_5)(\gamma_1 - i\gamma_2)\gamma_0 \tag{53}$$

$$e*^{(2)}(k) = \frac{1}{4}\,(1 + \gamma_5)(\gamma_1 + i\gamma_2)\gamma_0 \tag{54}$$

$$e*^{(3)}(k) = \frac{1}{2\sqrt{2}}\,(1 + \gamma_5)\gamma_3\gamma_0 \tag{55}$$

$$e*^{(0)}(k) = \frac{1}{2\sqrt{2}}\,(1 + \gamma_5) \tag{56}$$

For the non-axial vectors of the form (36) the terms $\frac{1}{2}\gamma_0\gamma_\mu$ constitute the vector basis. Indeed, we can use the relation

$$\mathrm{tr}[(\gamma_0\gamma_\mu)(\gamma_0\gamma_\nu)] = \mathrm{tr}[(\gamma^\mu\gamma_0)(\gamma_0\gamma_\nu)] = 4g^\mu{}_\nu \tag{57}$$

in order to project precisely those components; namely, we should seek

$$C^\mu_\Psi = \mathrm{tr}(\tfrac{1}{2}\gamma^\mu\gamma_0\Psi) \tag{58}$$

(γ_0 is included to account for the other γ_0 factor that is included in the solutions).

For solutions w_i, $\tilde{w}_i$ in Tables III and IV an orthonormal vector basis can be found in the vector interpretation of Eq. (46) which contains them. This is obtained by using, for example, the vectors

$$b^\mu = \gamma_0 \frac{i}{2\sqrt{-\Box}}(\gamma^\mu\partial\!\!\!/ + \sqrt{2}\partial^\mu) \tag{59}$$

which also satisfy $b^{\mu*} = \dfrac{-i}{2\sqrt{-\Box}}(\gamma^\mu - \sqrt{2}\partial^\mu)\gamma_0$,

$$\mathrm{tr}(b^*_\mu b_\nu) = g_{\mu\nu} \tag{60}$$

as can be shown by using the relation $\mathrm{tr}(\gamma_\mu\partial\!\!\!/\gamma_\nu\partial\!\!\!/) = 4(2\partial_\mu\partial_\nu - g_{\mu\nu}\Box)$. The presence of the ∂_μ in these expressions uses the fact that we may generate a vector solution from a scalar by taking the derivative. Given that we have constructed solutions that satisfy Dirac's equation (1), it follows that the solutions will also satisfy the Klein–Gordon equation. This means that when projecting the solutions on vectors (59) these can only be defined as a limiting case, as the $1/\sqrt{-\Box}$ operator is singular when applied to the solutions.

3.3. Gauge Invariance

A clue for the interpretation of all massless solutions described so far is suggested from the fact that Eq. (2) is invariant to first order under a set of gauge transformations, that is, with local dependence, which implies some are spurious. In trying to generate this transformation, we expect it to be unitary and Lorentz-invariant. However, we can only present a transformation satisfying either property, but not both together. We expect that it satisfies both properties when applied on the space of solutions. This is reminiscent of the QFT case.

We now consider the transformation $U_G = e^{iG}$ in the sense of Eq. (3) with generator $G = \partial\!\!\!/ a(x)$, and $a(x)$ any real function, where we use the form $H \to \tilde{U}^\dagger H\tilde{U}$. When applying the corresponding infinitesimal transformation to the operator $\gamma_0\partial\!\!\!/$ we need to consider only the commutator [or anticommutator, if we take $i\partial\!\!\!/ a(x)$ as the generator] with the Dirac operator, which contain

$$[\not{\partial}, \not{\partial} a(x)]_{\pm} = \Box a(x) \pm \partial_\mu a(x)\, \partial_\nu \gamma^\mu \gamma^\nu \tag{61}$$

The (anti)commutator with the operator $a(x)\, \not{\partial}$ gives

$$[\not{\partial}, a(x)\, \not{\partial}]_{\pm} = \partial_\mu a(x)\, \partial_\nu \gamma^\mu \gamma^\nu \pm a(x) \Box \tag{62}$$

From these equations we see that $U_{G^a} = e^{iG^a}$, where

$$G^a = \not{\partial} a(x) + a(x) \not{\partial} \tag{63}$$

(or the transformation with generator $\not{\partial} a(x) - a(x)\not{\partial}$) will be invariant to first order provided $\Box a(x) + 2\partial_\mu a(x) k^\mu = 0$. Consequently, the symmetry is linked to the space of solutions. We also get a cancellation to second order in G^a if $\Box a(x) = 0$ and $\partial_\mu a(x) k^\mu = 0$ are satisfied. We note that these conditions mean $a(x)$ satisfies the massless Klein–Gordon equation in a reduced number of directions. Although U_{G^a} is a Lorentz-invariant operator, it is not Hermitian.

The term

$$G_5^b = i[b(x)\gamma_5 \not{\partial} - \gamma_5\, \not{\partial} b(x)] \tag{64}$$

under the conditions $\Box b(x) = 0$ and $\partial_\mu b(x) k^\mu = 0$ is similarly the generator of another symmetry operator $U_{G_5^b} = e^{iG_5^b}$ since

$$[\not{\partial}, \gamma_5\, \not{\partial} b(x)]_{\pm} = \gamma_5(-\Box b(x) \pm \partial_\mu b(x)\, \partial_\nu \gamma^\mu \gamma^\nu) \tag{65}$$

and

$$[\not{\partial}, b(x)\gamma_5\, \not{\partial}]_{\pm} = \gamma_5(-\partial_\mu b(x)\, \partial_\nu \gamma^\mu \gamma^\nu \pm b(x) \Box) \tag{66}$$

We have obtained two sets of local transformations restrained by the condition on the functions $a(x)$ and $b(x)$. We may understand this as a manifestation of a gauge symmetry, where we attribute the restriction to the choice of gauge. Indeed, we find a similarity with the gauge invariance of the electromagnetic field A_μ. The Lorentz gauge condition (Lorentz-invariant) for it

$$\partial^\mu A_\mu = 0 \tag{67}$$

reduces Maxwell's equations to

$$\Box A_\mu = 0 \tag{68}$$

In this case the gauge freedom is reduced to transformations $A_\mu \rightarrow A_\mu + \partial_\mu \phi$, where ϕ satisfies $\Box \phi = 0$. The fact that solutions in Tables I–IV also satisfy the massless Klein–Gordon equation supports the interpretation of these solutions as vector particles satisfying Maxwell's equation within the Lorentz gauge. In fact, these solutions resemble more the case of the quantized electromagnetic field in this gauge. It is easy to see that this set of solutions does not satisfy Eq. (67). However, Eq. (2) can be interpreted as implying that the Lorentz condition is satisfied in the mean, a condition required for the quantized electromagnetic field

$$\partial^{\mu} A_{\mu} |\psi\rangle = 0 \tag{69}$$

Where $|\psi\rangle$ describes states from the electromagnetic field. To put Eq. (2) in this form, we only need to use combined solutions as obtained in Eq. (36).

It is interesting that in our case Eq. (69) is a condition that we derive and not one that we impose additionally from gauge fixing. We therefore obtain again a connection to QFT. The suggested gauge symmetry also would imply that not all the solutions in the fields in Tables III and IV are independent, but rather that they could be obtained from the fields in Tables I and II by a gauge transformation. The presence of a gauge symmetry places constraints on the choice of terms in a quatum relativistic equation, as happens in QFT. Thus, here we find a justification for the choice of Lorentz-invariant terms in Eq. (2).

3.4. Massive Solutions

In order to describe the solutions of Eq. (2) with $M \neq 0$ we choose the rest frame so that they only have time dependence

$$\Psi_{ki}^{(+)M}(x) = U_i(k)\, e^{-iMt} \tag{70}$$

$$\Psi_{ki}^{(-)M}(x) = V_i(k)\, e^{iMt} \tag{71}$$

The matrix components are classified by the eigenvalue of the parity operator P into the $P = -1$ group in Table V and the $P = 1$ group in Table VI. Here the $\tilde{0}$ subscript labels the solutions with negative eigenvalue of $\frac{i}{2}\gamma_1\gamma_2$. Where the solutions are classified with the aid of the normalized mass operator

Table V. Parity $P = -1$ Massive Bosons

Massive bosons	γ_0	$\frac{i}{2}\gamma_1\gamma_2$	$[H/k_0, \cdot]$	$[\mathbf{\Sigma} \cdot \hat{\mathbf{k}}, \cdot]$
$U_1(M, \mathbf{0}) = \frac{1}{4}(1 + \gamma_0)(\gamma_1 + i\gamma_2)$	1	1/2	2	1
$V_1(M, \mathbf{0}) = \frac{1}{4}(1 - \gamma_0)(\gamma_1 + i\gamma_2)$	−1	1/2	−2	1
$U_{-1}(M, \mathbf{0}) = \frac{1}{4}(1 + \gamma_0)(\gamma_1 - i\gamma_2)$	1	−1/2	2	−1
$V_{-1}(M, \mathbf{0}) = \frac{1}{4}(1 - \gamma_0)(\gamma_1 - i\gamma_2)$	−1	−1/2	−2	−1
$U_0(M, \mathbf{0}) = \frac{1}{4}(1 + \gamma_0)(\gamma_5 - \gamma_3)$	1	1/2	2	0
$V_0(M, \mathbf{0}) = \frac{1}{4}(1 - \gamma_0)(\gamma_5 + \gamma_3)$	−1	1/2	−2	0
$U_{\tilde{0}}(M, \mathbf{0}) = \frac{1}{4}(1 + \gamma_0)(\gamma_5 + \gamma_3)$	1	−1/2	2	0
$V_{\tilde{0}}(M, \mathbf{0}) = \frac{1}{4}(1 - \gamma_0)(\gamma_5 - \gamma_3)$	−1	−1/2	−2	0

Table VI. Parity $P = 1$ Massive Bosons

Massive bosons	γ_0	$\frac{i}{2}\gamma_1\gamma_2$	$[H/k_0, \cdot]$	$[\mathbf{\Sigma} \cdot \hat{\mathbf{k}}, \cdot]$
$\bar{U}_1(M, \mathbf{0}) = \frac{1}{4}\gamma_5(1 - \gamma_0)(\gamma_1 + i\gamma_2)$	1	1/2	0	1
$\bar{V}_1(M, \mathbf{0}) = \frac{1}{4}\gamma_5(1 + \gamma_0)(\gamma_1 + i\gamma_2)$	−1	1/2	0	1
$\bar{U}_{-1}(M, \mathbf{0}) = \frac{1}{4}\gamma_5(1 - \gamma_0)(\gamma_1 - i\gamma_2)$	1	−1/2	0	−1
$\bar{V}_{-1}(M, \mathbf{0}) = \frac{1}{4}\gamma_5(1 + \gamma_0)(\gamma_1 - i\gamma_2)$	−1	−1/2	0	−1
$\bar{U}_0(M, \mathbf{0}) = \frac{1}{4}\gamma_5(1 - \gamma_0)(\gamma_5 + \gamma_3)$	1	1/2	0	0
$\bar{V}_0(M, \mathbf{0}) = \frac{1}{4}\gamma_5(1 + \gamma_0)(\gamma_5 - \gamma_3)$	−1	1/2	0	0
$\bar{U}_{\tilde{0}}(M, \mathbf{0}) = \frac{1}{4}\gamma_5(1 - \gamma_0)(\gamma_5 - \gamma_3)$	1	−1/2	0	0
$\bar{V}_{\tilde{0}}(M, \mathbf{0}) = \frac{1}{4}\gamma_5(1 + \gamma_0)(\gamma_5 + \gamma_3)$	−1	−1/2	0	0

$H/M = \gamma_0$ and the helicity $\frac{i}{2}\gamma_1\gamma_2$ [this operator is obtained from the limiting case $|\mathbf{k}| \to 0$ in Eq. (18)]. We note that the solutions are mixed components of vector, antisymmetric, and scalar components. We can also construct combinations with definite properties under the discrete transformations. Thus we find vectors $\gamma_0\gamma_\mu$, axial vectors $\gamma_5\gamma_0\gamma_\mu$, scalars γ_0, and pseudoscalars $\gamma_5\gamma_0$. We obtain that the vectors become massive and their longitudinal components become physical. Just as in the massless case, an orthogonal polarization basis can be defined. For the pseudovector, its transverse and longitudinal components are not physical.

On the other hand, the condition that they all belong to a vector representation forces us to assume the antisymmetric and scalar terms are in fact derivatives as in the massless case. We note also there remain two vector components constructed without internal spin, that is, constructed from derivatives of scalar particles. This structure is reminiscent of the Higgs mechanism, in which massive vector fields absorb scalar degrees of freedom due to breaking of the symmetry.

4. MASSLESS SPIN-1/2 PARTICLES

We now show it is possible to give a Lorentz transformation which describes fermions, too. This is done more naturally in the context of a matrix equation of the type of Eq. (2) and whose solutions are bosons and fermions. This constitutes progress in the task of giving a unified description of these fields. Indeed, we obtain solutions that under Lorentz transformations of the form (3) one of the sides transforms trivially, and therefore we get spin-1/2 objects transforming as the (1/2, 0) or (0, 1/2) representations of the Lorentz group.

The equation

$$(1 - \gamma_5) i\gamma_0 \partial_\mu \gamma^\mu \Psi = 0 \tag{72}$$

has this type of solution. The invariance algebra of this equation contains the Lorentz generator

$$J^-_{\mu\nu} = \tfrac{1}{2}(1 - \gamma_5) J_{\mu\nu} = \tfrac{1}{2}(1 - \gamma_5)[i(x_\mu \partial_\nu - x_\nu \partial_\mu) + \tfrac{1}{2}\sigma_{\mu\nu}] \tag{73}$$

(and the other Poincaré generators). Among the solutions of Eq. (72) we have again the $V - A$ vectors u_i in Table I which under the effect of $\frac{1}{2}(1 - \gamma_5)\, \mathbf{\Sigma} \cdot \hat{\mathbf{k}}$ and $H/k_0 = \frac{1}{2}(1 - \gamma_5)\gamma_0\gamma^3$ lead to the same quantum numbers.

The Dirac operator in Eq. (72) is defined on a 2×2 matrix space; nevertheless, the solutions lie in the larger 4×4 matrix space. It is precisely this structure which leads to a set of solutions classified as spin-1/2 particles under $J^-_{\mu\nu}$. Actually, we have as additional symmetry of Eq. (72) the group of linear complex transformations $G(2, C)$ with eight components generated by

$$\tfrac{1}{2}(1 + \gamma_5), \qquad f_{\mu\nu} = -\tfrac{i}{2}(1 + \gamma_5)\sigma_{\mu\nu} \tag{74}$$

where $\sigma_{\mu\nu}$ is given in Eq. (15). This means Eq. (72) has a freedom in the choice of the Lorentz transformation since, e.g., both $J^-_{\mu\nu}$ and $J_{\mu\nu}$ are possible ones. The w_i terms in Table III are also a set of solutions of Eq. (72). However, their interpretation changes to fermions when using $J^-_{\mu\nu}$ to classify them. Clearly, the nature of the solutions depends on the Hamiltonian and the set of transformations that we choose to classify them. But once our choice is made, there is no ambiguity.

The unitary subgroups $SU(2) \times U(1)$ of the $G(2,C)$ symmetry operators in Eq. (74) imply we have two additional quantum numbers we can assign to the solutions. Considering that this symmetry does not act on the vector solution part, and taking account of the known quantum numbers of fermions in nature, we shall associate these operators with flavor and lepton number, respectively. The $SU(2)$ set of operators in (74) leads to a flavor doublet. The $U(1)$ is in this case not independent of the chirality. Choosing among the generators of $SU(2)$ f_{30} to classify the solutions of Eq. (72), these are given in Table VII where the caret is used to distinguish the flavor, and in

Table VII. Massless Fermions

Left-handed spin-1/2 particles	$\frac{1}{2}(1 - \gamma_5)\gamma_0\gamma^3$	$\frac{i}{4}(1 - \gamma_5)\gamma_1\gamma_2$	$[f_{30}, \cdot]$
$w_{-1/2}(k) = \frac{1}{4}(1 - \gamma_5)(\gamma_0 + \gamma_3)$	1	−1/2	1/2
$w_{-1/2}(\tilde{k}) = \frac{1}{4}(1 - \gamma_5)(\gamma_1 + i\gamma_2)$	−1	1/2	1/2
$\hat{w}_{-1/2}(k) = \frac{1}{4}(1 - \gamma_5)(\gamma_1 - i\gamma_2)$	1	−1/2	−1/2
$\hat{w}_{-1/2}(\tilde{k}) = \frac{1}{4}(1 - \gamma_5)(\gamma_0 - \gamma_3)$	−1	1/2	−1/2

this case the product and commutator of H and $\frac{1}{2}(1 - \gamma_5)\mathbf{\Sigma} \cdot \hat{\mathbf{k}}$ give the same results. Equation (39) can be used to obtain the full coordinate dependence. As in the Weyl equation, we obtain solutions of a defined chirality or helicity. We also have negative-energy solutions of the form (44) whose Hermitian conjugates are interpreted as right-handed antiparticles. The latter could have been obtained by departing from an equation with $V + A$ solutions. In order to have a Dirac fermion and a fermion mass, we need to have an equation mixing both chirality solutions. These shall be obtained in Sections 7 and 8.

4.1. Gauge Invariance

We prove Eq. (72) has a gauge symmetry in the limiting case of $\alpha_+ \rightarrow 0$ in

$$[\alpha_+ \tfrac{1}{2}(1 + \gamma_5) + \alpha_- \tfrac{1}{2}(1 - \gamma_5)] i\gamma_0 \partial_\mu \gamma^\mu \Psi = 0 \tag{75}$$

Using Eqs. (61)–(64) and commutation relations with $i\gamma_5\gamma_0\partial_\mu\gamma^\mu$, it can be shown that

$$[(1 - \gamma_5)\not\partial, (1 + \gamma_5)\not\partial b(x)]_- = 2[\not\partial, \not\partial b(x)]_- - 2\gamma_5[\not\partial, \not\partial b(x)]_+ \tag{76}$$

$$[(1 - \gamma_5)\not\partial, (1 + \gamma_5)b(x)\,\not\partial]_- = 2[\not\partial, b(x)\not\partial]_- - 2\gamma_5[\not\partial, b(x)\not\partial]_+ \tag{77}$$

$$[(1 + \gamma_5)\not\partial, (1 - \gamma_5)\not\partial b(x)]_- = 2[\not\partial, \not\partial b(x)]_- + 2\gamma_5[\not\partial, \not\partial b(x)]_+ \tag{78}$$

$$[(1 + \gamma_5)\not\partial, (1 - \gamma_5)b(x)\not\partial]_- = 2[\not\partial, b(x)\not\partial]_- + 2\gamma_5[\not\partial, b(x)\not\partial]_+ \tag{79}$$

The application of the symmetry generator

$$\overline{G}^b = [\alpha_-(1 + \gamma_5) + \alpha_+(1 - \gamma_5)]G^b \tag{80}$$

to Eq. (75), where G^a is given in Eq. (63), produces the terms in Eqs. (76)–(79) and cancels the anticommutator contributions. Therefore, we obtain that $\overline{G}^b$ will be a generator of a gauge symmetry of Eq. (75) in the sense explained before. In fact, in the limit $\alpha_+ \rightarrow 0$ the symmetry is satisfied to first order since all other terms in the exponential $\exp(i\overline{G}^6)$ cancel. Unlike the case of Eq. (25), we note that only one gauge symmetry is allowed.

5. CURRENTS AND VERTEX INTERACTION

From Noether's theorem we have a conserved current for each continuous symmetry present in the system. We can construct, using Eqs. (2) and (6), bilinear current vector operators j_μ based on Eq. (4) and current vector expectation values $\mathrm{tr}(j_\mu)$ based on Eq. (5) satisfying

$$\partial^\mu j_\mu = 0, \qquad \partial^\mu \, \mathrm{tr}(j_\mu) = 0 \tag{81}$$

The form of the j_μ is similar to the currents in Dirac's equation, given that some symmetries are shared by the Dirac equation and Eqs. (2), (6). In the case of Eqs. (2), (25), and (72) we also have the global symmetry $\Psi \to e^{ia}\Psi$, where a is a real parameter. The corresponding current operator is

$$j_\mu = \Psi^\dagger \gamma_0 \gamma_\mu \Psi \tag{82}$$

This symmetry implies the conservation of number of particles, with the zero component of the current being positive definite, so that it can be interpreted as a probability density. This component has already been considered when setting a normalization condition in Eq. (30).

The symmetry $e^{ib\gamma_5}$, with b real, is valid for the massless equations (25) and (72) and leads to the chirality current

$$j_\mu{}^5 = \Psi^\dagger \gamma_5 \gamma_0 \gamma_\mu \Psi \tag{83}$$

Expressions can be obtained also for the currents corresponding to the energy-momentum tensor and the generalized angular momentum which are equal to those obtained for the Dirac equations. It is these which underlie the classification of the solutions with H and $\mathbf{\Sigma} \cdot \hat{\mathbf{k}}$. This partly justifies as well the classification done with the commutators of these operators and the solutions, given that they are also eigenfunctions under them.

The current operator corresponding to the gauge symmetry in Eq. (75) (which overlaps with the above currents j_μ, $j_\mu{}^5$) is given by

$$j_\mu^{\mathrm{gau}} = \Psi'^\dagger \tfrac{1}{2}(1 - \gamma_5)\gamma_0\gamma_\mu \Psi \tag{84}$$

(we take here the bra $\Psi'^\dagger$ possibly distinct from the ket Ψ). Comparison of the current j_μ^{gau} with the form of the field

$$A_\mu^-(k) = \frac{i}{2\sqrt{2}}\,(1 - \gamma_5)\gamma_0\gamma_\mu \, e^{-ikx} \tag{85}$$

derived from the u_i terms in Table I strongly suggests a connection to the transition matrix element of the A_μ^- operator field between the two massless fermion solutions Ψ' and Ψ,

$$\langle \Psi' | A_\mu^-(k) | \Psi \rangle \tag{86}$$

Indeed, in QFT the minimal coupling $\mathcal{L} = g j_\mu^{\mathrm{gau}} A^{-\mu}$ in the Lagrangian implies a vertex interaction, which can lead to the expectation value of the form

$$g[\tfrac{1}{2}(1 - \gamma_5)\gamma_0\gamma_\mu]_{\alpha\beta} \to g\langle u(p_f, s_f) | \tfrac{1}{2}(1 - \gamma_5)\gamma_0\gamma_\mu | u(p_i, s_i)\rangle \tag{87}$$

where $(p_{i,f}, s_{i,f})$ are the initial and final momenta and spins of the fermions,

k_μ is the momentum of the vector field, and g is the coupling constant. A consistent interpretation of Eqs. (84)–(86) is possible along these lines by understanding $\langle\Psi'|A_\mu^-(k)|\Psi\rangle$ as an interaction with its assignment to the vertex in Eq. (87) and the coupling constant $g = 1/\sqrt{2}$.

Note that a more formal argument should take account of the exponential factor in Eq. (85), which can be done in the context of QFT; it would lead, together with the spatial dependence of the fermion wave functions, to Dirac's delta $(2\pi)^4\delta^4(k - p_f + p_i)$. Also, the substitution (87) is one of many ways to obtain a contribution in a perturbation expansion in terms of diagrams. Although $A_\mu^-(k)$ should be properly normalized as a field of units of [energy], it is enough for our argument to keep the polarization normalized.

6. LORENTZ (3, 1) STRUCTURE AND SCALARS FROM SIX-DIMENSIONAL CLIFFORD ALGEBRA

In previous sections we derived a description of fermions and bosons through equations implied by the structure of the Clifford algebra in $D = 4$. Although the structure obtained is too simple to describe thoroughly aspects of the SM (for example, the model cannot include massive fermions), we have useful results which we would like to keep as the prediction of interactions in the form of vertices relating vectors and fermions, coupling constants, and, in particular, hints of a description of isospin on left-handed particles. These features are expected to remain in higher dimensions, where we find a more elaborate structure.

The simplest generalization of the above model is to consider the six-dimensional Clifford algebra (the $D = 5$ case lives also in a 4×4 space). This is composed of 64 8×8 matrices and it can be obtained as a tensor product of the original 4×4 algebra and the 2×2 matrices generated by the unit matrix 1_2 and the three Pauli matrices σ_1, σ_2, σ_3. We will use a basis for the 8×8 matrix space in which we can identify the underlying $D = 4$ components

$$\begin{aligned} \gamma_0 \to \gamma_0' &= 1_2 \otimes \gamma_0, \qquad \gamma_1 \to \gamma_1' = 1_2 \otimes \gamma_1, \\ \gamma_2 \to \gamma_2' &= \sigma_1 \otimes \gamma_2, \qquad \gamma_3 \to \gamma_3' = 1_2 \otimes \gamma_3 \end{aligned} \tag{88}$$

Then

$$1_8, \quad \sigma_1 \otimes 1_4, \quad \gamma_5' = \sigma_2 \otimes \gamma_2, \quad \gamma_6' = \sigma_3 \otimes \gamma_2 \tag{89}$$

are 4D scalars since they commute with the spin operators

$$\sigma'_{\mu\nu} = \frac{i}{2}[\gamma'_\mu, \gamma'_\nu], \qquad \mu = 0, \ldots, 3, \quad \nu = 0, \ldots, 3 \tag{90}$$

In fact, the matrices γ'_μ defined in Eqs. (88) and (89) form the 6D Clifford algebra

$$\{\gamma'_\mu, \gamma'_\nu\} = 2g_{\mu\nu} \tag{91}$$

As all γ_μ are generalized to 8×8 matrices through a tensor product $\gamma_\mu \to 1_2 \otimes \gamma_\mu$, $\mu = 0, \ldots, 3$, without danger of ambiguity we shall use a notation in which we now assume that γ_μ represent 8×8 matrices. We also use the quaternion-like notation for the representation of 1_2 and the Pauli matrices in the 8×8 matrix space,

$$1_8 = 1_2 \otimes 1_4, \qquad I = \sigma_1 \otimes 1_4, \qquad J = \sigma_2 \otimes 1_4, \qquad K = \sigma_3 \otimes 1_4 \tag{92}$$

The 4D algebra will be written in terms of

$$\gamma'_\mu = \gamma_\mu, \qquad \mu = 0, 1, 3, \quad \gamma'_2 = I\gamma_2 \tag{93}$$

and the scalars in Eq. (89) (here in Hermitian form) in terms of

$$1 = 1_8, \qquad I, \quad i\gamma'_5 = iJ\gamma_2, \quad i\gamma'_6 = iK\gamma_2 \tag{94}$$

Because I, J, K commute with γ_2, it is possible to omit the tensor product sign. A more explicit form of these matrices can be found in the Appendix. Then all 64 elements of the 8×8 algebra are obtained by multiplying the 16 elements of the 4D algebra generated by the terms in Eq. (93) and the elements of (94) and they can be written with this notation. Hence, it will be possible to identify every element constructed in this way in terms of the 4D Lorentz representation to which it belongs.

The preceding definitions will also be applied for the assignment $\gamma_5 \to 1_2 \otimes \gamma_5 \equiv \gamma_5$. Then, besides the scalar elements of Eq. (89) [or Eq. (94)] we have the scalars

$$\gamma_5, \quad I\gamma_5, \quad J\gamma_2\gamma_5, K\gamma_2\gamma_5 \tag{95}$$

$I\gamma_5$ commutes with these and the scalar elements in Eq. (94). Excluding it and the identity, the remaining six elements generate an $SO(4)$ algebra, or equivalently, an $SU(2) \times SU(2)$ algebra. The latter's generators consist of the right-handed elements

$$\frac{1}{4}(1 + I\gamma_5)I, \quad \frac{i}{4}(1 + I\gamma_5)J\gamma_2, \quad \frac{i}{4}(1 + I\gamma_5)K\gamma_2 \tag{96}$$

and left-handed elements

$$I_1 = \frac{i}{4}(1 - I\gamma_5)J\gamma_2 \tag{97}$$

$$I_2 = -\frac{i}{4}(1 - I\gamma_5)K\gamma_2 \tag{98}$$

$$I_3 = -\frac{1}{4}(1 - I\gamma_5)I \tag{99}$$

The eight form an $SU(2)_R \times SU(2)_L \times U(1) \times U(1)$ algebra, where the subscripts L and R are added accordingly (the normalization is chosen to fit $\frac{1}{2}\sigma_i$).

6.1. Chain Breaking of $D = 6$ Algebra

The above symmetry operators immediately show a close connection to the actual symmetries observed in nature at the massless level, that is, the $SU(2)_L$ of isospin and $U(1)_Y$ of hypercharge groups. The eight scalars in Eqs. (94) and (95) have a Cartan algebra of dimension four, for which we can take the basis 1, I, γ_5, $I\gamma_5$. These operators can be arranged into the four projection operators

$$P_{++} = \tfrac{1}{4}(1 + I\gamma_5)(1 + I) \tag{100}$$

$$P_{+-} = \tfrac{1}{4}(1 + I\gamma_5)(1 - I) \tag{101}$$

$$P_{-+} = \tfrac{1}{4}(1 - I\gamma_5)(1 + I) \tag{102}$$

$$P_{--} = \tfrac{1}{4}(1 - I\gamma_5)(1 - I) \tag{103}$$

which, when combined with the Dirac operator, create the general massless Lorentz-invariant equation

$$(\alpha_{++}P_{++} + \alpha_{+-}P_{+-} + \alpha_{-+}P_{-+} + \alpha_{--}P_{--})\gamma_0(i\partial^\mu\gamma'_\mu)\Psi = 0, \quad \mu = 0, \ldots, 3 \tag{104}$$

We then have four different Lorentz-invariant degrees of freedom α_{++}, α_{+-}, α_{-+}, α_{--} for constructing a generalized equation. One or various vanishing coefficients lead to degrees of freedom disappearing from the spectrum. In fact, the choice of nonvanishing coefficients divides this equation into four classes. For each class we assume that all fields transform under the same Lorentz representation. Additional conditions on the coefficients might lead to more symmetries to appear. The different choices are as follows.

In class I, only one coefficient is nonvanishing, e.g., $\alpha_{-+} \neq 0$, and $\alpha_{++} = \alpha_{+-} = \alpha_{--} = 0$ (we will not consider the different four permutations of the α_{ij} belonging to this class and others, which have similar properties).

Without loss of generality here and in similar cases, we may assume $\alpha_{-+} = 1$. This type of equation is similar to Eq. (72) except that in this case, in addition to the $U(1)$ gauge symmetry generated by P_{-+}, we have a flavor $SU(6)$, whose elements are projected by $P_{+-} + P_{-+} + P_{--}$.

In class II, in which two α_{ij} vanish, we have in general at least a $U(1)$ gauge symmetry and a $SU(4)$ flavor symmetry. Furthermore, we consider three possibilities for choices of the α_{ij}. In the case $\alpha_{-+} = \alpha_{--} \neq 0$ we have in particular a $U(1)_L \times SU(2)_L$ gauge symmetry. In this case, both fermions and vectors are obtained in the spectrum, but the fermions are all left-handed as for solutions in Table VII and their antiparticles right-handed. The cases $\alpha_{++} \neq 0$, $\alpha_{-+} \neq 0$ or $\alpha_{++} \neq 0$, $\alpha_{--} \neq 0$ resemble Eq. (75) and break any possible gauge $SU(2)$ symmetry.

For class III, where only one $\alpha_{ij} = 0$, we have in general an $SU(2)$ flavor symmetry and three gauge $U(1)$'s. In the case, $\alpha_{-+} = \alpha_{--}$ instead of one $U(1)$, we have also a gauge $SU(2)_L$ symmetry; by setting, e.g., $\alpha_{-+} = \alpha_{+-}$ or $\alpha_{--} = \alpha_{+-}$ we get an equation which has a projection of the form of Eq. (25), that is, with parity as a symmetry, a condition necessary to have a solution of the form of an electromagnetic field. The representations contain both vectors and fermions which are both left-handed and right-handed.

Finally, for class IV, in the case $\alpha_{++} = \alpha_{+-} = \alpha_{--} = \alpha_{-+}$ we have a gauge $U(2)_L \times U(2)_R$ and the representations only contain vectors of the type appearing in Tables I–IV. There is a possibility of finding a similar description to that of class III if we use a Lorentz transformation projected with $L = P_{+-} + P_{-+} + P_{--}$. This case will not be considered.

Of the four choices described, it is type III (or type IV under the condition stated) with $\alpha_{-+} = \alpha_{--} = \alpha_{+-}$ which can be parity conserving and which contains an $SU(2)_L$ symmetry. We shall associate this group with the isospin and one $U(1)$ with the hypercharge. This case is analyzed in detail in the following section.

7. MASSLESS CASE: TYPE III SPECTRUM, UNIFIED $SU(2) \times U(1)$

We analyze the equation

$$iL\gamma_0\partial^\mu\gamma'_\mu\Psi = 0, \qquad \mu = 0, \ldots, 3 \tag{105}$$

where we use the projection operator

$$L = P_{+-} + P_{-+} + P_{--} = \tfrac{3}{4} - \tfrac{1}{4}(I + \gamma_5 + I\gamma_5) \tag{106}$$

which corresponds to the type III case with $\alpha_{+-} = \alpha_{-+} = \alpha_{--}$. The equation is invariant under the set of Lorentz transformations

$$J^L_{\mu\nu} = L[i(x_\mu \partial_\nu - x_\nu \partial_\mu) + \tfrac{1}{2}\sigma'_{\mu\nu}] \tag{107}$$

where $\sigma'_{\mu\nu}$ is defined in Eq. (90). The scalar symmetries are classified into flavor, with its generators projected by P_{++}, an $SU(2)_L$ gauge symmetry, and two other $U(1)$ gauges, according to the arguments following Eq. (75). We choose one generator of $SU(2)_L$ to classify the solutions, say, I_3 in Eq. (99), with eigenvalue I_{s3}. The other two $U(1)$ gauge generators are chosen orthogonal to I_3 and will be taken

$$Y = -1 + \tfrac{1}{2}(I + \gamma_5) \tag{108}$$

and $\tilde{\gamma}_5 = L\gamma_5$. Here $\tilde{\gamma}_5$ is orthogonal to Y and I_3 in the sense of $\mathrm{tr}(\tilde{\gamma}_5 Y) = 0$, and $\mathrm{tr}(\tilde{\gamma}_5 I_3) = 0$. The choice in Eq. (108) can be obtained from the demand that the operator lead to a gauge symmetry in the sense of Eq. (80). Another justification for these definitions will be given later. Although we call it gauge symmetry, we still need to prove that a chiral symmetry like the I_i actually is, but we shall make this assumption. There are also global symmetries which are related to particle number conservation. L is interpreted as the lepton number, whose quantum number we denote by l. The Casimir of $SU(2)_L$, $I_1^2 + I_2^2 + I_3^2$, with eigenvalue $I_s(I_s + 1)$, is not an independent component, but a linear combination of $\tilde{\gamma}_5$ and Y.

We present the fermion and boson solutions of Eq. (105), which we classify according to the Hamiltonian and helicity projections $H = L\gamma_0\gamma^3$, $\frac{i}{2}LI\gamma_1\gamma_2$, I_3, and the quantum numbers Y, I_s. We also define the flavor as the generator $f_{30} = \frac{1}{2}(1 - L)\gamma_3\gamma_0$ and its eigenvalue f.

7.1. Spin-1/2 particles

The $l = 1$, $I_s = 1/2$, $Y = -1$ massless fermions are given in Table VIII, where the subscript $-1/2$ refers to the value of the helicity operator $[\mathbf{\Sigma} \cdot \hat{\mathbf{p}}, \cdot]$, so that we also present particles with opposite momentum, and the index L denotes (redundantly) the left-handed character of the solutions. The space-time dependence of these solutions can be obtained also following Eq. (39). Negative-energy solutions are obtained by changing the sign of the exponential, and the Hermitian conjugates of the latter are the antiparticle solutions.

These spin-1/2 particles belong to the fundamental representation of the non-Abelian group $SU(2)_L$ and are labeled also by the Y operator. Considering the quantum numbers of leptons in nature, it follows we can associate Y with the hypercharge and the I_i with the three generators of isospin. We also associate the two elements distinguished only by the f quantum number (and a caret) with a flavor doublet, which we identify with any two lepton families among the three generations, e.g., the left-handed electron and muon and their neutrinos.

Table VIII. $l = 1$, $I_s = {}^1/_2$, $Y = -1$ Massless Fermion Multiplets in 5 + 1 D

Left-handed massless spin-1/2 particles $f = 1/2$	$L\gamma_0\gamma^3$	$\frac{i}{2}LI\gamma_1\gamma_2$	I_3
$\begin{pmatrix} \nu_{-1/2}(k) \\ l^-_{-1/2}(k) \end{pmatrix}_L = \begin{pmatrix} \frac{1}{8}(1 - I\gamma_5)(J\gamma_2 - iK\gamma_2)(\gamma_0 + \gamma_3) \\ \frac{1}{8}(1 - I\gamma_5)(1 + I)(\gamma_0 + \gamma_3) \end{pmatrix}$	1	−1/2	1/2 −1/2
$\begin{pmatrix} \nu_{-1/2}(\tilde{k}) \\ l^-_{-1/2}(\tilde{k}) \end{pmatrix}_L = \begin{pmatrix} \frac{1}{8}(1 - I\gamma_5)(J\gamma_2 - iK\gamma_2)(\gamma_1 + iI\gamma_2) \\ \frac{1}{8}(1 - I\gamma_5)(1 + I)(\gamma_1 + iI\gamma_2) \end{pmatrix}$	−1	1/2	1/2 −1/2
Left-handed massless spin-1/2 particles $f = -1/2$	**$L\gamma_0\gamma^3$**	**$\frac{i}{2}LI\gamma_1\gamma^2$**	**I_3**
$\begin{pmatrix} \hat{\nu}_{-1/2}(k) \\ \hat{l}^-_{-1/2}(k) \end{pmatrix}_L = \begin{pmatrix} \frac{1}{8}(1 - I\gamma_5)(J\gamma_2 - iK\gamma_2)(\gamma_1 - iI\gamma_2) \\ \frac{1}{8}(1 - I\gamma_5)(1 + I)(\gamma_1 - iI\gamma_2) \end{pmatrix}$	1	−1/2	1/2 −1/2
$\begin{pmatrix} \hat{\nu}_{-1/2}(\tilde{k}) \\ \hat{l}^-_{-1/2}(\tilde{k}) \end{pmatrix}_L = \begin{pmatrix} \frac{1}{8}(1 - I\gamma_5)(J\gamma_2 - iK\gamma_2)(\gamma_0 - \gamma_3) \\ \frac{1}{8}(1 - I\gamma_5)(1 + I)(\gamma_0 - \gamma_3) \end{pmatrix}$	−1	1/2	1/2 −1/2

Another part of the spectrum has positive chirality, $l = 1$, $I_s = 0$, $Y = -2$, and is given in Table IX, where antiparticles can be obtained with the corresponding transformations, and as in previous cases, the solutions presented can be obtained from each other by a rotation. The quantum numbers correspond to right-handed charged leptons, as we will show, again in good correspondence with the SM.

7.2. Vectors

The pure vector solutions are similar to the u_i, $\tilde{u}_i$ terms in Tables I and II. The isospin scalars can be separated into their $V + A$ and $V - A$ components (all have lepton number $l = 0$, as required). The first are given in Table X. Comparing these solutions with those in Table II, we see they differ by the substitution (93) and the projector P_{+-}. Similarly, the $V - A$ terms can be obtained straightforwardly from Table I and the projector $P_{-+} + P_{--}$. These are given in Table XI, Taking account of the normalization, a combination of the terms B_i and $\tilde{B}_i$ can be taken which carries the hypercharge Y in Eq.

Table IX. $l = 1$, $I_s = 0$, $Y = -2$, Massless Fermion Multiplet in 5 + 1 D.

Right-handed massless spin-1/2 particles	$L\gamma_0\gamma^3$	$\frac{i}{2}LI\gamma_1\gamma_2$	$[f_{30}]_i$
$l^-_{1/2R}(k) = \frac{1}{8}(1 + I\gamma_5)(J\gamma_2 - iK\gamma_2)\gamma_0(\gamma_1 + iI\gamma_2)$	1	1/2	1/2
$l^-_{1/2R}(\tilde{k}) = \frac{1}{8}(1 + I\gamma_5)(J\gamma_2 - iK\gamma_2)\gamma_0(\gamma_0 + \gamma_3)$	−1	−1/2	1/2
$\hat{l}^-_{1/2R}(k) = \frac{1}{8}(1 + I\gamma_5)(J\gamma_2 - iK\gamma_2)\gamma_0(\gamma_0 - \gamma_3)$	1	1/2	−1/2
$\hat{l}^-_{1/2R}(\tilde{k}) = \frac{1}{8}(1 + I\gamma_5)(J\gamma_2 - iK\gamma_2)\gamma_0(\gamma_1 - iI\gamma_2)$	−1	−1/2	−1/2

Table X. $I_s = 0$, $Y = 0$, $V + A$ Vectors in 5 + 1 D

Vector solutions	$[H/k_0, \cdot]$	$[\mathbf{\Sigma} \cdot \hat{\mathbf{p}}, \cdot]$
$\tilde{B}_1(k) = \frac{1}{8}(1 + I\gamma_5)(1 - I)\gamma_0(\gamma_1 + iI\gamma_2)$	2	1
$\tilde{B}_1(\tilde{k}) = \frac{1}{8}(1 + I\gamma_5)(1 - I)\gamma_0(\gamma_1 - iI\gamma_2)$	2	1
$\tilde{B}_0(k) = \frac{1}{8}(1 + I\gamma_5)(1 - I)\gamma_0(\gamma_0 - \gamma_3)$	0	0
$\tilde{B}_0(\tilde{k}) = \frac{1}{8}(1 + I\gamma_5)(1 - I)\gamma_0(\gamma_0 + \gamma_3)$	0	0

(108). We shall associate this combination with the B_μ fields which carry the hypercharge in the Weinberg–Salam model [5,7,8].

Three additional sets of solutions of the equation can be described in terms of the fields B_i in Table XI and the generators of $SU(2)_L$ in Eqs. (97)–(99), which are written in Table XII. As these $V - A$ vector solutions belong to the adjoint representation of group $SU(2)_L$, $I_s = 1$, we associate them with the fields $W^{\pm}_\mu$, W^0_μ of the electroweak theory.

7.3. Scalars and Antisymmetric Tensors

The last part of the boson spectrum is composed of scalars and antisymmetric $Y = -1$ doublets (and antiparticles). The solutions are constructed similarly to the $\tilde{w}_i$ components in Table IV with the addition of the factors containing I, $J\gamma_2$, $K\gamma_2$, which account for the hypercharge and isospin quantum numbers. The corresponding $Y = -1$ doublets are given in Table XIII, the same problems arise regarding the Lorentz interpretation of antisymmetric terms as for Tables III and IV. The same procedure can be used in extracting vector and scalar components from these solutions. Again here is a parallelism with the SM. A scalar particle appears in a doublet and we will see that it is involved in giving masses to the particles. For this reason, we may associate this degree of freedom with the Higgs particle. We leave open the question

Table XI. $I_s = 0$, $Y = 0$, $V - A$ Vectors in 5 + 1 D

Vector solutions	$[H/k_0, \cdot]$	$[\mathbf{\Sigma} \cdot \hat{\mathbf{p}}, \cdot]$
$B_{-1}(k) = \frac{1}{4\sqrt{2}}(1 - I\gamma_5)\gamma_0(\gamma_1 - iI\gamma_2)$	2	−1
$B_{-1}(\tilde{k}) = \frac{1}{4\sqrt{2}}(1 - I\gamma_5)\gamma_0(\gamma_1 + iI\gamma_2)$	2	−1
$B_0(k) = \frac{1}{4\sqrt{2}}(1 - I\gamma_5)\gamma_0(\gamma_0 - \gamma_3)$	0	0
$B_0(\tilde{k}) = \frac{1}{4\sqrt{2}}(1 - I\gamma_5)\gamma_0(\gamma_0 + \gamma_3)$	0	0

Table XII. Isospin Triplet Vector Bosons in 5 + 1 D

Isospin vector carriers	I_3
$W_i^+(k) = \frac{1}{\sqrt{2}}(J\gamma_2 - iK\gamma_2)B_i(k)$	1
$W_i^0(k) = IB_i(k)$	0
$W_i^-(k) = \frac{1}{\sqrt{2}}(J\gamma_2 + iK\gamma_2)B_i(k)$	−1

of whether these mass terms can be obtained from a gauge transformation, although the form of the proposed gauge transformation here suggests it should be possible.

Summarizing, the positive-energy solutions are the vectors B_i and $\tilde{B}_i$, which amount to eight degrees of freedom, where we are taking account of both directions of momenta for given helicity. The isospin vectors $W_i^{\pm,0}$ have 12 degrees of freedom and the antisymmetric tensors and scalars $\tilde{n}_i$ and $\tilde{\nu}_i$ have 8, and with their antiparticles, 16. These add up to 36 bosons. We have obtained massless spin-1/2 particles in a doublet and a singlet; these use 4 and 2 degrees of freedom, respectively. Taking account of antiparticles and the two flavors, we have 24 fermion degrees of freedom. Altogether, these add up to 60 degrees of freedom. The reason for not having 64 active is the 4 inert degrees of freedom projected by P_{++}, which are not influenced by the Hamiltonian, projected by L.

8. MASSIVE CASE: SYMMETRY BREAKING OF $SU(2) \times U(1)$

In seeking a massive extension of Eq. (105) we expect all the Hermitian combinations of the scalar terms in Eqs. (94) and (95), multiplied by γ_0

Table XIII. $I_s = {}^1/_2$, $Y = -1$ Boson Chiral Terms in 5 + 1 D

Scalars and antisymmetric tensors	$[H/k_0, \cdot]$	$[\boldsymbol{\Sigma} \cdot \hat{\mathbf{p}}, \cdot]$	I_3
$\begin{pmatrix}\tilde{n}_0(k)\\ \tilde{\nu}_0(k)\end{pmatrix} = \begin{pmatrix}\frac{1}{8}(1+I\gamma_5)(J\gamma_2 - iK\gamma_2)(\gamma_0+\gamma_3)\\ \frac{1}{8}(1+I\gamma_5)(1-I)(\gamma_0+\gamma_3)\end{pmatrix}$	2	0	1/2 −1/2
$\begin{pmatrix}\tilde{n}_0(\tilde{k})\\ \tilde{\nu}_0(\tilde{k})\end{pmatrix} = \begin{pmatrix}\frac{1}{8}(1+I\gamma_5)(J\gamma_2 - iK\gamma_2)(\gamma_0-\gamma_3)\\ \frac{1}{8}(1+I\gamma_5)(1-I)(\gamma_0-\gamma_3)\end{pmatrix}$	2	0	1/2 −1/2
$\begin{pmatrix}\tilde{n}_1(k)\\ \tilde{\nu}_1(k)\end{pmatrix} = \begin{pmatrix}\frac{1}{8}(1+I\gamma_5)(J\gamma_2 - iK\gamma_2)(\gamma_1+iI\gamma_2)\\ \frac{1}{8}(1+I\gamma_5)(1-I)(\gamma_1+iI\gamma_2)\end{pmatrix}$	0	1	1/2 −1/2
$\begin{pmatrix}\tilde{n}_1(\tilde{k})\\ \tilde{\nu}_1(\tilde{k})\end{pmatrix} = \begin{pmatrix}\frac{1}{8}(1+I\gamma_5)(J\gamma_2 - iK\gamma_2)(\gamma_1-iI\gamma_2)\\ \frac{1}{8}(1+I\gamma_5)(1-I)(\gamma_1-iI\gamma_2)\end{pmatrix}$	0	1	1/2 −1/2

(in a Hamiltonian form of the equation), to be scalars with respect to the Lorentz transformation

$$J'_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) + \tfrac{1}{2}\sigma'_{\mu\nu} \tag{109}$$

which just generalizes Eq. (14). However, if we also demand that they be scalars with respect to $J^L_{\mu\nu}$ in Eq. (107), then the choices are reduced to

$$M_1 = \frac{M}{2}(1 - I) \tag{110}$$

$$M_2 = \frac{iM}{2}(\gamma_5 - I\gamma_5) \tag{111}$$

$$M_3 = \frac{M}{2} J\gamma_2(-1 + \gamma_5) \tag{112}$$

$$M_4 = \frac{M}{2} K\gamma_2(-1 + \gamma_5) \tag{113}$$

where M is the mass constant. Now, the only nontrivial scalar that commutes with all M_i terms is L. Nevertheless, if we relax this condition, we obtain in addition that precisely and only

$$Q = I_3 + \tfrac{1}{2}Y \tag{114}$$

commutes with M_3 and M_4 ($Q' = I_3 - \frac{1}{2}Y$ commutes with M_1 and M_2). As Q is the electric charge, we deduce the electromagnetic $U(1)_{em}$ remains a symmetry while the hypercharge and isospin are broken. We stress that Q is deduced, rather than being imposed, as the only additional symmetry consistent with massive terms. M_3 and M_4 do not commute among themselves, but can be obtained from each other through a unitary transformation involving γ_5. We therefore choose one, M_3, to study the massive representations. We will show the equation

$$(L\gamma_0 i\partial^\mu \gamma'_\mu - M_3\gamma_0)\Psi = 0, \qquad \mu = 0, \ldots, 3 \tag{115}$$

gives rise to massive and massless fermions and vectors that are contained in the SM, at symmetry breaking.

8.1. Vectors

Despite the presence of a massive term, we get a set of vector components which remain massless, as their product with the mass term M_3 (or M_4) in Eq. (115) vanishes. These are the combination of the massless terms in Tables XI and XII,

$$A_{Li} = \frac{1}{\sqrt{2}}\,(B_i - W_i^0) \tag{116}$$

There are several parity operators for Eq. (115) with the necessary properties. They differ by the square, which leads to different projection operator combinations. The only one leading to nontrivial solutions acts on the same space as Q and is of the same rank. This is

$$P = M_3\gamma_0\mathcal{P} \tag{117}$$

where $\mathcal{P}$ is defined as for Eq. (34) and M_3 is given in Eq. (112).

The remaining bosons become massive. The massive chargeless solutions are a combination of the vectors B_i, $\tilde{B}_i$ in Tables X and XI, the W_i^0 in Table XII, the $\tilde{n}_i$ bosons in Table XIII, and their antiparticles n_i with $n_0(\tilde{k}) = \tilde{n}_0^\dagger(\tilde{k})$, $n_0 = \tilde{n}^\dagger(k)$, $n_{-1}(k) = -\tilde{n}_1^\dagger(k)$, $n_{-1}(\tilde{k}) = -\tilde{n}_1^\dagger(\tilde{k})$. We construct the latter using Table III and multiplying on the left by operators carrying the isospin and hypercharge, with phases as in Table XIII. The solutions can be classified in two groups, depending on the value of the commutator $[M_3\gamma_0, \Psi]$, or, equivalently, by the value of the operator P. When the commutator is zero we have the solutions in Table XIV:

Table XIV. $P = 1$ Massive Bosons

Massive bosons	$M_3\gamma_0/M$	$\frac{i}{2}LI\gamma_1\gamma_2$	$[H/k_0, \cdot]$	$[\mathbf{\Sigma}\cdot\hat{\mathbf{k}}, \cdot]$
$P_1(M, \mathbf{0}) = \frac{1}{2}\,(\tilde{B}_1(k) + \frac{1}{\sqrt{2}}\,(B_{-1}(\tilde{k}) + W^0_{-1}(\tilde{k})) - \tilde{n}_1(k) + n_{-1}(\tilde{k}))$	1	1/2	0	1
$Q_1(M, \mathbf{0}) = \frac{1}{2}\,(\tilde{B}_1(k) + \frac{1}{\sqrt{2}}\,(B_{-1}(\tilde{k}) + W^0_{-1}(\tilde{k})) + \tilde{n}_1(k) - n_{-1}(\tilde{k}))$	−1	1/2	0	1
$P_{-1}(M, \mathbf{0}) = \frac{1}{2}\,(\tilde{B}_1(\tilde{k}) + \frac{1}{\sqrt{2}}\,(B_{-1}(k) + W^0_{-1}(k)) - \tilde{n}_1(\tilde{k}) + n_{-1}(k))$	1	−1/2	0	−1
$Q_{-1}(M, \mathbf{0}) = \frac{1}{2}\,(\tilde{B}_1(\tilde{k}) + \frac{1}{\sqrt{2}}\,(B_{-1}(k) + W^0_{-1}(k)) + \tilde{n}_1(\tilde{k}) - n_{-1}(k))$	−1	−1/2	0	−1
$P_0(M, \mathbf{0}) = \frac{1}{2}\,(\tilde{B}_0(k) + \frac{1}{\sqrt{2}}\,(B_0(\tilde{k}) + W^0_0(\tilde{k})) - \tilde{n}_0(\tilde{k}) - n_0(\tilde{k}))$	1	1/2	0	0
$Q_0(M, \mathbf{0}) = \frac{1}{2}\,(\tilde{B}_0(k) + \frac{1}{\sqrt{2}}\,(B_0(\tilde{k}) + W^0_0(\tilde{k})) + \tilde{n}_0(k) + n_0(\tilde{k}))$	−1	1/2	0	0
$P_{\tilde{0}}(M, \mathbf{0}) = \frac{1}{2}\,(\tilde{B}_0(\tilde{k}) + \frac{1}{\sqrt{2}}\,(B_0(k) + W^0_0(k)) - \tilde{n}_0(\tilde{k}) - n_0(k))$	1	−1/2	0	0
$Q_{\tilde{0}}(M, \mathbf{0}) = \frac{1}{2}\,(\tilde{B}_0(\tilde{k}) + \frac{1}{\sqrt{2}}\,(B_0(k) + W^0_0(k)) + \tilde{n}_0(\tilde{k}) + n_0(k))$	−1	−1/2	0	0

The k and $\tilde{k}$ arguments simply label the vector components (in the massless solutions) in terms of which the massive solutions are constructed. The nonzero terms for the commutator with $M_3\gamma_0$ are given in Table XV:

Table XV. $P = -1$ Massive Bosons

Massive bosons	$M_3\gamma_0/M$	$\frac{i}{2}LI\gamma_1\gamma_2$	$[H/k_0, \cdot]$	$[\mathbf{\Sigma} \cdot \hat{\mathbf{k}}, \cdot]$
$\overline{P}_1(M, \mathbf{0}) = \frac{1}{2}(\tilde{B}_1(k)) - \frac{1}{\sqrt{2}}(B_{-1}(\tilde{k}) + W^0_{-1}(\tilde{k})) + \tilde{n}_1(k) + n_{-1}(\tilde{k}))$	1	1/2	2	1
$\overline{Q}_1(M, \mathbf{0}) = \frac{1}{2}(\tilde{B}_1(k) - \frac{1}{\sqrt{2}}(B_{-1}(\tilde{k}) + W^0_{-1}(\tilde{k})) - \tilde{n}_1(k) - n_{-1}(\tilde{k}))$	−1	1/2	−2	1
$\overline{P}_{-1}(M, \mathbf{0}) = \frac{1}{2}(\tilde{B}_1(\tilde{k}) - \frac{1}{\sqrt{2}}(B_{-1}(k) + W^0_{-1}(k)) + \tilde{n}_1(\tilde{k}) + n_{-1}(k))$	1	−1/2	2	−1
$\overline{Q}_{-1}(M, \mathbf{0}) = \frac{1}{2}(\tilde{B}_1(\tilde{k}) - \frac{1}{\sqrt{2}}(B_{-1}(k) + W^0_{-1}(k)) - \tilde{n}_1(\tilde{k}) - n_{-1}(k))$	−1	−1/2	−2	−1
$\overline{P}_0(M, \mathbf{0}) = \frac{1}{2}(\tilde{B}_0(k) - \frac{1}{\sqrt{2}}(B_0(\tilde{k}) + W^0_0(\tilde{k})) + \tilde{n}_0(k) - n_0(\tilde{k}))$	1	1/2	2	0
$\overline{Q}_0(M, \mathbf{0}) = \frac{1}{2}(\tilde{B}_0(k) - \frac{1}{\sqrt{2}}(B_0(\tilde{k}) + W^0_0(\tilde{k})) - \tilde{n}_0(k) + n_0(\tilde{k}))$	−1	1/2	−2	0
$\overline{P}_{\tilde{0}}(M, \mathbf{0}) = \frac{1}{2}(\tilde{B}_0(\tilde{k}) - \frac{1}{\sqrt{2}}(B_0(k) + W^0_0(k)) + \tilde{n}_0(\tilde{k}) - n_0(k))$	1	−1/2	2	0
$\overline{Q}_{\tilde{0}}(M, \mathbf{0}) = \frac{1}{2}(\tilde{B}_0(\tilde{k}) - \frac{1}{\sqrt{2}}(B_0(k) + W^0_0(k)) - \tilde{n}_0(\tilde{k}) + n_0(k))$	−1	−1/2	−2	0

We have also a set of charged vector particles, constructed from the $W^\pm_\mu$ in Table XII and the charged doublet components $\tilde{\nu}_i$ in Table XIII, and their antiparticles. The $Q = -1$ components are given in Table XVI:

Table XVI. Charged $Q = -1$ Massive Vector Bosons

Massive W's	$M_3\gamma_0/M$	$\frac{i}{2}LI\gamma_1\gamma_2$	$[H/k_0, \cdot]$	$[\mathbf{\Sigma} \cdot \hat{\mathbf{k}}, \cdot]$
$W^-_{M1} = \frac{1}{\sqrt{2}}(W^-_{-1}(\tilde{k}) - \tilde{\nu}_1(k))$	1	1/2	1	1
$\hat{W}^-_{M1} = \frac{1}{\sqrt{2}}(W^-_{-1}(\tilde{k}) + \tilde{\nu}_1(k))$	−1	1/2	−1	1
$W^-_{M-1} = \frac{1}{\sqrt{2}}(W^-_{-1}(k) - \tilde{\nu}_1(\tilde{k}))$	1	−1/2	1	−1
$\hat{W}^-_{M-1} = \frac{1}{\sqrt{2}}(W^+_{-1}(k) + \tilde{\nu}_1(\tilde{k}))$	−1	−1/2	−1	−1
$W^-_{M0} = \frac{1}{\sqrt{2}}(W^-_0(\tilde{k}) - \tilde{\nu}_0(k))$	1	1/2	1	0
$\hat{W}^-_{M0} = \frac{1}{\sqrt{2}}(W^-_0(\tilde{k}) + \tilde{\nu}_0(k))$	−1	1/2	−1	0
$W^-_{M\tilde{0}} = \frac{1}{\sqrt{2}}(W^-_0(k) + \tilde{\nu}_0(\tilde{k}))$	1	−1/2	1	0
$\hat{W}^-_{M\tilde{0}} = \frac{1}{\sqrt{2}}(W^-_0(k) - \tilde{\nu}_0(\tilde{k}))$	−1	−1/2	−1	0

where the $\tilde{0}$ subscript labels the solution with negative eigenvalue of $\frac{i}{2}LI\gamma_1\gamma_2$. The positively charged terms can be obtained from $(W^-_{\tilde{M}i})^\dagger$.

8.2. Spin-1/2 Particles

The application of the massive terms M_3 and M_4 in Eqs. (112) and (113) to the $Y = -1$ $I_{s3} = 1/2$ (with $Q = 0$) "neutrino" elements and their antiparticles gives zero, which implies the neutrinos remain massless. In addition, the neutrino and antineutrino solutions lack a right-handed and left-handed partner, respectively, to be able to form Dirac massive particles.

On the other hand, the massive term M_3 (or M_4) breaks the chiral symmetry mixing values of chirality and causing the charged fermions to acquire a mass. Their lepton number $l = 1$ and charge $Q = -1$ wave functions are given in Table XVII:

Table XVII. Charged $Q = -1$ Massive Fermions

Charged massive spin-1/2 particles	$[M_3\gamma_0/M, \cdot]$	$[\frac{i}{2}LI\gamma_1\gamma_2, \cdot]$	$[f_{3i}]$
$u^-_{1/2} = \frac{1}{\sqrt{2}}(l^-_{-1/2L}(\tilde{k}) - l^-_{1/2R}(k))$	1	1/2	1/2
$v^-_{1/2} = \frac{1}{\sqrt{2}}(l^-_{-1/2L}(\tilde{k}) + l^-_{1/2R}(k))$	−1	1/2	1/2
$u^-_{-1/2} = \frac{1}{\sqrt{2}}(l^-_{-1/2L}(k) - l^-_{1/2R}(\tilde{k}))$	1	−1/2	1/2
$v^-_{-1/2} = \frac{1}{\sqrt{2}}(l^-_{-1/2L}(k) + l^-_{1/2R}(\tilde{k}))$	−1	−1/2	1/2
$\hat{u}^-_{1/2} = \frac{1}{\sqrt{2}}(\hat{l}^-_{-1/2L}(\tilde{k}) - \hat{l}^-_{1/2R}(k))$	1	1/2	−1/2
$\hat{v}^-_{1/2} = \frac{1}{\sqrt{2}}(\hat{l}^-_{-1/2L}(\tilde{k}) + \hat{l}^-_{1/2R}(k))$	−1	1/2	−1/2
$\hat{u}^-_{-1/2} = \frac{1}{\sqrt{2}}(\hat{l}^-_{-1/2L}(k) - \hat{l}^-_{1/2R}(\tilde{k}))$	1	−1/2	−1/2
$\hat{v}^-_{-1/2} = \frac{1}{\sqrt{2}}(\hat{l}^-_{-1/2L}(k) + \tilde{l}^-_{1/2R}(\tilde{k}))$	−1	−1/2	−1/2

The charge of these fermions leads to their association with any two of the negatively charged massive leptons e^-, μ^-, or τ^-.

9. RELATION TO PHYSICAL FIELDS

There remains to classify the vector fields obtained in the breaking of the $SU(2)_L \times U(1)_Y$ to the Q symmetry, according to the discrete symmetries. The terms found, A_{Li} in Eq. (116) and P_i, $\overline{P}_i$, Q_i, $\overline{Q}_i$ in Tables XIV and XV, sum to 20 degrees of freedom. Similar combinations as for the massive vector terms U_i, $\bar{U}_i$, V_i, $\bar{V}_i$ in Tables V and VI can be taken to obtain terms with the necessary transformation properties.

The (normalized) vector component solutions of Tables XIV and XV, which transform as a non-axial vector by P in Eq. (117), are given by

$$A_\mu = \frac{1}{2} Q\gamma_0\gamma_\mu \tag{118}$$

A_μ can be represented as a mixture of two chargeless and massless components. On one hand, Tables XIV and XV contain the special combination of the B_i in Table X and the $\tilde{B}_i$ in Table XI, which form precisely

$$B_\mu = \frac{1}{2\sqrt{3}} Y\gamma_0\gamma_\mu \tag{119}$$

that is, the hypercharge carriers. This gives another justification for the choice of Y given in Eq. (108), which is the operator giving the correct values for the hypercharge of fermions. Thus we obtain another argument needed to set Y, whose background is in the way we arrive at the expression for Q in Eq. (114). On the other hand, we can extract the chargeless vector components for the isospin triplet in Table XII,

$$W^0_\mu = I_3\gamma_0\gamma_\mu \tag{120}$$

where I_3 is given in Eq. (99).

From the expression for Q and Eqs. (118)–(120) we easily obtain

$$A_\mu = \frac{1}{2} W^0_\mu + \frac{\sqrt{3}}{2} B_\mu \tag{121}$$

The value of the Weinberg angle θ_W is derived immediately from this equation by making an analogy with the new fields obtained in the SM after application of the Higgs mechanism. The photon then has the form

$$A_\mu = \frac{1}{\sqrt{g'^2 + g^2}} (gB_\mu + g'W^0_\mu) \tag{122}$$

where g and g' are, respectively, the isospin and hypercharge coupling constants. We obtain $g'/g = 1/\sqrt{3}$. As in the SM $\tan(\theta_W) = g'/g$, we find

$$\sin^2(\theta_W) = 0.25 \tag{123}$$

The Z_μ field can constructed by considering the orthogonal combination to A_μ in Eq. (121),

$$Z_\mu = \frac{\sqrt{3}}{2} W^0_\mu - \frac{1}{2} B_\mu \tag{124}$$

We therefore find the A_μ and Z_μ span the vector components in Tables XIV and XV.

The charged massive solutions can be related to the $W^\pm_\mu$ components in the SM. Although we obtained a difference in the masses of the Z_μ and $W^\pm_\mu$, this does not correspond to the one obtained in the SM. Also, the vector particle A_μ is massive. We attribute these differences to the fact that term $M_3\gamma_0$ does not commute with the kinetic term in Eq. (105), which is required to allow for simultaneously massive and massless solutions in the space projected by Q. Indeed, the space spanned by the vectors A_{Li} in Eq. (116) is annihilated by M_3. This fact allows them to be massless solutions.

9.1. Coupling Constants: Vector Fermion–Current Vertices

Following the steps which allow for a vertex interpretation of Eq. (86), it is possible to derive the vertices describing the coupling of the fermions to the vectors obtained from the solutions. This information can be summarized through the Lagrangian density

$$\begin{aligned}\mathcal{L} = {} & \frac{g}{2\sqrt{2}} [\nu^\dagger(1 - I\gamma_5)\gamma_0\gamma^\mu W^+_\mu + hc] \\ & - \frac{e}{2} [\tan(\theta_W)(2l^\dagger_R\gamma_0\gamma^\mu l_R + \nu^\dagger\gamma_0\gamma^\mu\nu + l^\dagger_L\gamma_0\gamma^\mu l_L) \\ & + \cot(\theta_W)(\nu^\dagger\gamma_0\gamma^\mu\nu - l^\dagger_L\gamma_0\gamma^\mu l_L)]Z_\mu - eu^\dagger\gamma_0\gamma^\mu u A_\mu \end{aligned} \tag{125}$$

where ν, l_L are given in Table VIII, l_R are given in Table IX, and u is given in Table XVII. The electric charge is given by $e = gg'/\sqrt{g'^2 + g^2}$.

In addition, the vertices give information on the coupling constants g' and g, which cannot be extracted from Eqs. (121) or (124). Information on these can be obtained by calculating the overlap of the vectors with the corresponding fermion currents, which are given explicitly in Eq. (125). This can be done more realistically by considering the (7 + 1)-dimensional Clifford algebra, where vector massless solutions become possible. The coupling constant g can be obtained from the coupling of the massive charged vectors $W^+_{M\mu}$ in Table XVI and the charged current obtained from the neutrino and the charged massive lepton wave functions, represented by the first term of Eq. (125). We obtain g as

$$g = 1/\sqrt{2} \approx 0.707 \tag{126}$$

The coupling g' is deduced from the second term in Eq. (125) to be

$$g' = 1/\sqrt{6} \approx 0.408 \tag{127}$$

It is a consistency check of the theory that these values agree with the Weinberg angle in Eq. (123). (At $5 + 1$ dimensions we also obtain the consistent values $g = 1$, $g' = 1/\sqrt{3} \approx 0.577$). In addition, these values are to be compared with the experimentally measured ones at energies of the mass of the W particle, which is where the breakdown of the $SU(2)_L \times U(1)_Y$ symmetry occurs. These are $g'_{\text{exp}} \approx 0.35$, $g_{\text{exp}} \approx 0.65$, and $\sin^2(\theta_{\text{Wexp}}) \approx 0.23$.

10. SUMMARY AND CONCLUSIONS

In this work we departed from a generalized Dirac equation whose solutions, with the rules we have postulated to interpret them, exhibit some similarity to quantum fields. We first studied these in the framework of the $3 + 1$ Clifford matrices. They comprise non-axial and axial vectors, spin-zero particles, antisymmetric tensors, and, under a choice of the Lorentz generators, even spin-1/2 particles of a given chirality. We investigated a gauge symmetry of the equation. A comparison among the different solutions is possible with the application of a generalized point product within the quantum mechanical framework of the equation. Through it the transition amplitude of a vector field and two fermions is a vertex, and hence it is interpreted as an interaction. The coupling constant is then determined.

We also investigated the simplest generalization of the equation, which is in the context of a $(5 + 1)$-dimensional Clifford algebra. By focusing on the $3 + 1$ underlying structure we obtained an $SU(2)_L \times U(1)$ symmetry. We get a boson and fermion set of solutions for the massless case. The addition of a mass term to the equation implies the breaking of the symmetry to $U(1)_Q$, which can be interpreted as the gauge symmetry defining the electromagnetic interaction. We also obtain the field solutions, their spectrum, and some of the couplings among them. We showed they exhibit a close similarity to the particles and coupling constants in the $SU(2)_L \times U(1)$ sector of the standard model at symmetry breaking.

The main result in this work has been to derive gauge interactions and the particle spectrum from an extended spin space, some of whose components transform under the usual Lorentz generators in $3 + 1$ space-time. The gauge forces emerge as excitations determined by the symmetries permitted by the Clifford algebra in which the $3 + 1$ subalgebra is embedded. Thus, we find a relation between gauge and space-time symmetries. In the simplest Clifford algebra containing the $3 + 1$ subalgebra, we found a symmetry as large as $U(2)_L \times U(2)_R$ and we showed we have only two choices for a model with an $SU(2)_L \times U(1)$ which contains both fermions and bosons. The $SU(2)_L \times U(1)_L$ symmetry group is consequently derived rather than being imposed.

It is noteworthy that the chiral nature of the $SU(2)$ gauge interaction is predicted. The formalism also predicts gauge vector carriers which are also generators lying in the adjoint representation of the group.

In general, a field theory is determined by the couplings among fields, which are defined at tree level. The power of field theory in describing nature stems from the possibility of using this simple description in perturbation theory to account for more complex behavior by considering repeated interactions. The values of the coupling constants are arbitrary and must be fixed by experiment. In our case, the very nature of the fermion and boson solutions defines the coupling at tree level. In fact, our theory determines the type of fields involved and the normalization restriction fixes the values of the coupling constants. It is the compositeness feature of the solutions, the fact that some may be constructed from the product of others, that determines their interaction. For example, the form of the spin-1/2 particle pair coupling to vectors and scalar particles is restrained by the symmetries of the theory. Thus, the restrictiveness in the choice of the representations in our theory constitutes also its asset.

In the model described in Eq. (105) we obtained leptons with the correct gauge quantum numbers corresponding to a left-handed doublet of $SU(2)$ (that is, in the fundamental representation) with hypercharge $Y = -1$, and a singlet with $Y = -2$, which can be interpreted as massless neutrinos and charged leptons. These fields appear in doublets characterized by a conserved quantum number which does not affect interactions with vector bosons and which we have therefore associated with flavor. The flavor doublets are a consequence of using a Hamiltonian which allows for a certain matrix solution space, although the size of the flavor multiplet can change in higher dimensional models. This may constitute a clue to the puzzle of generations. Furthermore, the fermions have a conserved lepton number. Thus the fermions obtained could be identified with any pairs of the particle set e, ν_e; μ, ν_μ; and τ, ν_τ.

We also obtained a spinless boson doublet with $Y = -1$ which can be identified with a Higgs particle. It is interesting that this boson appears here as part of the solution representations and not put in by hand. We find that introducing a mass term into the equation, as represented by Eq. (115), implies an additional interaction of the scalar particle which gives masses to some of the fields. We showed that the $SU(2)_L \times U(1)_Y$ symmetry is broken to a $U(1)_Q$ symmetry. This procedure goes further than the SM, where the Higgs mechanism is a mathematical device to create massive terms, and which requires explicitly that $U(1)_Q$ remain unbroken. In our case the presence of a mass term implies that $U(1)_Q$ is the unbroken gauge symmetry in the real world.

We obtained masses for the vector bosons different from those in the SM. We have ascribed this difference to the fact that the 6-D model does not allow for massless vector solutions, which is permitted in the next Clifford algebra at $7 + 1$ dimensions. It is encouraging that the values obtained then for the coupling contants are within 7–15% of their values in the SM at electroweak breakdown. The agreement of the values of the coupling constants, vertices, and particles described in the theory is further fortified by the fact that other reducible representations will reproduce only some aspects, while others will change. For example, the trace, which fixes the interaction, is representation dependent.

There remain several aspects to be studied. The argument leading to the quantization condition in Eq. (69) implies that the present equations carry with them an implied gauge fixing. While we have shown in detail the extent to which this is true in the Abelian case, we still have to prove this for the non-Abelian case. Analogy with the non-Abelian QFT description requires ghosts to satisfy unitarity. These are scalar objects lying in the adjoint representation which do not appear as physical particles. We speculate spurious degrees of freedom as the antisymmetric n and ν could conform such a counterpart in extended theories. We have a different characterization of the spinless and Z_μ, W_μ fields from the corresponding ones in the SM in relation to their discrete transformation properties, since we get different weights for their scalar or psudoscalar and V, A contributions, respectively. As the Z_μ, W_μ particles interact only weakly and this characteristic fits into their interaction scheme, this aspect is, however, difficult to test. Furthermore, the interactions among vectors need further study. We attribute the fact that the mass of the lepton is of the same magnitude as for the W_μ to the unified approach we use. The possibility that this theory can provide information on the corrections to lepton masses is under investigation.

Finally, we heuristically obtained fields which reproduce properties of quantized operators in a quantum field theory. The implication is that quantization is not derived as a condition on the fields, but as a consequence of the definitions of the equations. This points to a closer relation to quantization which should be more researched more fully in the future. Causality and unitarity requirements demand more study on related aspects such as propagators, commutation relations, and how to include radiative corrections.

The presence of boson and fermion solutions became possible from the use of bispinors as solution space. Further extensions with the use of more spin indices will allow for a description of spin-3/2 and spin-2 objects; this may point to a connection to gravity. This possibility can be used in turn to propose a new interpretation of the wave function, with the implication of a closer connection of it to space-time. The development of this idea is done from another standpoint elsewhere [2].

The similarity in the representation of the fields in this formalism and the operators which carry out a Lorentz transformation for the spin parts could imply a possible connection between the two. Thus, a Lorentz transformation could be considered not independent of the fields needed to perform it physically. On the other hand, just as the choice of gauge interaction is restricted by the Clifford algebras, we also find that the interactions restrain the possible type of space-time symmetry. In this way we obtain a possible clue to the origin of the number of dimensions of space-time. Thus, although its (3, 1) structure is not predicted, it is among the few which are consistent with an $SU(2)_L \times U(1)$ symmetry in the $(5 + 1)$-dimensional Clifford algebra.

The unified treatment of space-time and gauge symmetries proposed here has proven fruitful. The formalism presented gives information on a set of representation solutions and their interactions by literally restricting them. Their agreement with aspects of the standard model makes the theory a plausible alternative, all the more so in that it assumes a rather conventional relativistic quantum mechanical framework of proven simplicity and universality. Information on additional aspects of the standard model may be found with the application of the theory in extended spaces, making certainly worth further study.

APPENDIX

We give here the conventions for the Clifford algebra used in this work and present explicitly the matrices generating it.

In four dimensions we use the metric

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{128}$$

The 4×4 matrices in the text are in the Dirac representation, and in order to define them we use the Pauli matrices σ_i, $i = 1, 2, 3$, and the 2×2 unit matrix 1_2:

$$\gamma_0 = \sigma_3 \otimes 1_2 = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix} \tag{129}$$

so the vector $\boldsymbol{\gamma} = (\gamma^1, \gamma^2, \gamma^3)$ is given by

$$\boldsymbol{\gamma} = i\sigma_2 \otimes \boldsymbol{\sigma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix} \tag{130}$$

All other matrices can be defined from these. For example,

$$\gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3 = \sigma_1 \otimes 1_2 = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix} \tag{131}$$

For the 6D Clifford algebra we use

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \tag{132}$$

The definitions leading to Eqs. (93) and (94) imply the 4D vector subset of 8×8 matrices are given explicitly by

$$\gamma'_\mu = 1_2 \otimes \gamma_\mu = \begin{pmatrix} \gamma_\mu & 0 \\ 0 & \gamma_\mu \end{pmatrix}, \qquad \mu = 0, 1, 3 \tag{133}$$

$$\gamma'_2 = \sigma_1 \otimes \gamma_2 = \begin{pmatrix} 0 & \gamma_2 \\ \gamma_2 & 0 \end{pmatrix} \tag{134}$$

and the 4D scalars by

$$1_8 = 1_2 \otimes 1_4 = \begin{pmatrix} 1_4 & 0 \\ 0 & 1_4 \end{pmatrix} \tag{135}$$

$$I = \sigma_1 \otimes 1_4 = \begin{pmatrix} 0 & 1_4 \\ 1_4 & 0 \end{pmatrix} \tag{136}$$

$$i\gamma'_5 = iJ\gamma_2 = i\sigma_2 \otimes \gamma_2 = \begin{pmatrix} 0 & \gamma_2 \\ -\gamma_2 & 0 \end{pmatrix} \tag{137}$$

$$i\gamma'_6 = iK\gamma_2 = i\sigma_3 \otimes \gamma_2 = \begin{pmatrix} i\gamma_2 & 0 \\ 0 & -i\gamma_2 \end{pmatrix} \tag{138}$$

All 8×8 matrices can be generated by products of these matrices. We use a notation in which the γ'_μ matrices are written in terms of the γ_μ matrices, and from Eq. (93) onwards the latter are assumed to be 8×8 matrices.

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